# Assignment 6

Due date: Thursday, October 12

*Ground state wavefunction of the electromagnetic field*

Recall from lecture the Hamiltonian operator for the electromagnetic field,

$$\hat{H} = \int d^3x \left( -\frac{(\hbar c)^2}{2} \partial_{A_i(\mathbf{x})} \partial_{A_i(\mathbf{x})} + \frac{1}{2} |\nabla \times \mathbf{A}(\mathbf{x})|^2 \right),$$

and ground state wave-functional:

$$\Psi_0[\mathbf{A}(\mathbf{x})] = \exp \left( -\kappa \int d^3x \int d^3y \, \frac{\nabla \times \mathbf{A}(\mathbf{x}) \cdot \nabla \times \mathbf{A}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \right).$$

Your task is to determine the value of the constant $\kappa$ that is consistent with the Schrödinger equation

$$\hat{H}\Psi_0 = E_0 \Psi_0.$$

You can do this without having to evaluate $E_0$ (which is infinite in the absence of a cutoff).

As a warm-up, let's do the analogous calculation for the 1D harmonic oscillator with Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \partial_x^2 + \frac{k}{2} x^2$$

and ground state wavefunction $\Psi_0 = \exp(-cx^2)$. The pair of derivatives in $\hat{H}$ acting on $\Psi_0$ produces two terms:

$$\partial_x^2 \Psi_0 = \left( \partial_x(-cx^2) \right)^2 \Psi_0 + \left( \partial_x \partial_x(-cx^2) \right) \Psi_0.$$

For the correct value of $c$, the first term exactly cancels the $(k/2)x^2\Psi_0$ term in $\hat{H}\Psi_0$. This is the extent of what you are being asked to do in the case of the electromagnetic field Schrödinger equation. It's enough to note that the second term is independent of $x$ (or $\mathbf{A}(\mathbf{x})$) and therefore its value (even if infinite!) is the energy eigenvalue.

For many of you the main challenge in this problem will be working with the "variational" or "functional" derivative $\partial_{A_i(\mathbf{x})}$. Below are some instances of how this operator acts. Be sure you understand each example! Repeated latin indices are summed, as usual.

In these, $f(\,)$ is an arbitrary scalar function (not functional) taking a vector argument:

$$\partial_{A_i(\mathbf{x})}\partial_{A_i(\mathbf{x})}\ f\left(\mathbf{A}(\mathbf{y})\right) = 0, \qquad \mathbf{x} \neq \mathbf{y}$$

$$\partial_{A_i(\mathbf{x})}\partial_{A_i(\mathbf{x})}\ f\left(\mathbf{A}(\mathbf{x})\right) = \nabla^2_{\mathbf{w}} f(\mathbf{w})\Big|_{\mathbf{w}=\mathbf{A}(\mathbf{x})}.$$

The following show the action of just a single functional derivative:

$$\partial_{A_i(\mathbf{x})}\left(A_j(\mathbf{x})A_j(\mathbf{y})\right) = A_i(\mathbf{y})$$

$$\partial_{A_i(\mathbf{x})}\int d^3y\ B_j(\mathbf{y})A_j(\mathbf{y}) = B_i(\mathbf{x})$$

$$\begin{aligned}\partial_{A_i(\mathbf{x})}\int d^3y\ f(\mathbf{y})\nabla\cdot\mathbf{A}(\mathbf{y}) &= \partial_{A_i(\mathbf{x})}\int d^3y\ \left(-\nabla f(\mathbf{y})\right)\cdot\mathbf{A}(\mathbf{y}) \\ &= -\partial_{x_i} f(\mathbf{x}).\end{aligned}$$

As in these examples, you may always neglect boundary terms when integrating by parts.

Here are some concrete hints, roughly in the order you'll need them.

1. $$\nabla_{\mathbf{z}}\ f(|\mathbf{x}-\mathbf{z}|) = -\nabla_{\mathbf{x}}\ f(|\mathbf{x}-\mathbf{z}|)$$

2. $$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

3. $$\int \frac{d^3z}{|\mathbf{x}-\mathbf{z}|^2|\mathbf{y}-\mathbf{z}|^2} = \frac{\pi^3}{|\mathbf{x}-\mathbf{y}|}$$

4. $$\nabla\cdot\nabla\times\mathbf{v} = 0$$

5. $$-\nabla^2_{\mathbf{x}}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right) = 4\pi\delta^3(\mathbf{x}-\mathbf{y})$$

Finally, here are some motivational quotes:

*Never give up, never give up, never give up!* — W. Churchill

*Integrate by parts, integrate by parts, integrate by parts!* — J. D. Jackson