Water surface wave potential energy

1. The potential energy of a volume element dV with density ρ at height z is given by $\rho gzdV$. The total potential energy is then given by $\Delta U_g = \int_0^{L_y}\int_0^{\lambda}\int_0^{s(x)} \rho gzdzdxdy = \frac{1}{2}\rho g L_y \int_0^{\lambda} s(x)^2 dx = \frac{1}{2}\rho g h^2 L_y \int_0^{\lambda} \cos^2(kx)\,dx = \frac{1}{4}\rho g h^2 L_y \int_0^{\lambda}(1+\cos(2kx))dx$. The second term in the integrand vanishes, since $\cos(2kx)$ oscillates around zero, with the positive and negative areas cancelling out. The gravitational potential energy is then $\Delta U = \frac{1}{4}\rho g h^2 L_y L_x = \frac{1}{4}\rho g h^2 A$. With this we finally have
$$\frac{\Delta U_g}{A} = \frac{1}{4}\rho g h^2.$$
2. Let the new area be denoted A' and the equilibrium area A_0, then $A' = \int_0^{L_y}\int_0^{\lambda}\sqrt{1+\left(\frac{ds}{dx}\right)^2}\,dxdy \approx L_y\int_0^{\lambda}\left(1+\frac{1}{2}\left(\frac{ds}{dx}\right)^2\right)dx$, and $A_0 = L_y\int_0^{\lambda}dx$. The change in area is then $\Delta A = A' - A_0 = \frac{1}{2}L_y\int_0^{\lambda}\left(\frac{ds}{dx}\right)^2 dx = \frac{1}{2}L_y h^2 k^2\int_0^{\lambda}\sin^2(kx)\,dx = \frac{1}{4}L_y h^2 k^2\int_0^{\lambda} 1 - \cos(2kx)\,dx$. As in part 1, the second term in the integrand vanishes, and we have
$$\Delta A = \frac{1}{4}h^2k^2L_xL_y = \frac{1}{4}h^2k^2A.$$
3. The surface energy per unit area is
$$\Delta U_\sigma = \frac{\sigma\Delta A}{A} = \frac{1}{4}\sigma h^2 k^2.$$

Motion of fluid elements in water-surface waves

1. We know $v_z(x, z=0) = \dot{s}(x,t)$ using the "small h" approximation. Therefore, $\dot{s}(x,t) = -h\sin(\omega t)\cos(kx)$, and therefore
$$s(x,t) = \frac{h}{\omega}\cos(\omega t)\cos(kx) = \frac{h}{\omega}(\cos(kx+\omega t)+\cos(kx-\omega t)).$$
For the case of the running water, we have $\dot{s}(x,t) = h(-\sin(\omega t)\cos(kx) + \cos(\omega t)\sin(kx))$, which means
$$s(x,t) = \frac{h}{\omega}(\cos(\omega t)\cos(kx)+\sin(\omega t)\sin(kx)) = \frac{h}{\omega}\cos(kx-\omega t).$$
2. The position is simply the time integral of the velocity, thus we have for the standing water surface
$$p_x = \int_{t_0}^{t} v_x dt' = h\sin(kx)\,e^{kz}\int_{t_0}^{t}\sin(\omega t)\,dt = -\frac{h}{\omega}\cos(\omega t)\sin(kx)\,e^{kz} + x$$
$$p_z = \int_{t_0}^{t} v_z dt' = -h\cos(kx)\,e^{kz}\int_{t_0}^{t}\sin(\omega t)\;dt' = \frac{h}{\omega}\cos(\omega t)\cos(kx)\,e^{kz} + z$$
Where we have chosen t_0 such that the average of p_x, p_z is x, z. For the case of running water, we have
$$p_x = \int_{t_0}^{t} h(\sin(\omega t)\sin(kx) + \cos(\omega t)\cos(kx))e^{kz}dt$$
$$= \frac{h}{\omega}e^{kz}(-\cos(\omega t)\sin(kx) + \sin(\omega t)\cos(kx)) + z = \frac{h}{\omega}e^{kz}\sin(\omega t - kx) + x$$

$$p_z = he^{kz}\int_{t_0}^{t} -\sin(\omega t)\cos(kx) + \cos(\omega t)\sin(kx)\ dt$$

$$= \frac{h}{\omega}e^{kz}(\cos(\omega t)\cos(kx) + \sin(\omega t)\sin(kx)) + z = \frac{h}{\omega}e^{kz}\cos(\omega t - kx) + z.$$

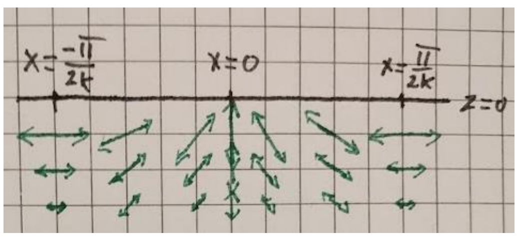
3. For plotting, it is convenient to write the above equations in vector form. For the standing case, we have

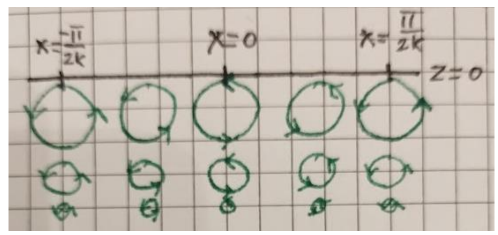
$$\begin{pmatrix} p_x \\ p_z \end{pmatrix} = \frac{h}{\omega}e^{kz}\cos(\omega t)\begin{pmatrix} \sin(kx) \\ \cos(kx) \end{pmatrix} + \begin{pmatrix} x \\ z \end{pmatrix}.$$

We see that in this case the time dependence is a periodic function multiplying both coordinates, implying linear oscillations around (x, z). For the running case, we have

$$\begin{pmatrix} p_x \\ p_z \end{pmatrix} = \frac{h}{\omega}e^{kz}\begin{pmatrix} \sin(\omega t - kx) \\ \cos(\omega t - kx) \end{pmatrix} + \begin{pmatrix} x \\ z \end{pmatrix}.$$

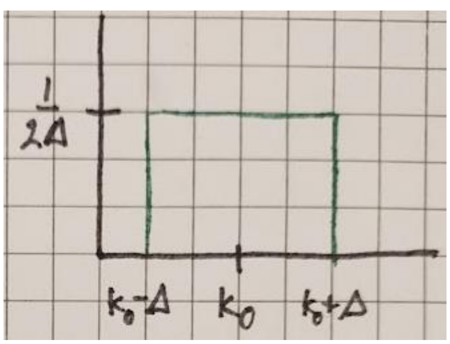
This is the parametric form for the equation of a circle with center (x, z), so in this case we have circular motion. Below are the sketches for the standing (left) and running (right) cases.





Rectangular wave packets

1.



2. The integral can be computed as follows

$$\int_{-\infty}^{\infty} h(k)\cos(kx)\,dk = \frac{1}{2\Delta}\int_{k_0-\Delta}^{k_0+\Delta}\cos\,(kx)dk = \frac{1}{2\Delta x}\left(\sin\left((k_0+\Delta)x\right) - \sin\left((k_0-\Delta)x\right)\right)$$

3. Using the sum and difference of angles trigonometric identities, we see that $\sin\left((k_0+\Delta)x\right) - \sin\left((k_0-\Delta)x\right) = 2\sin(\Delta x)\cos(k_0 x)$, so we can write

$$\int_{-\infty}^{\infty} h(k)\cos(kx)\,dk = \frac{\sin(\Delta x)}{\Delta x}\cos(k_0 x)$$

This is the product of an infinite oscillatory function, $\cos(k_0 x)$, and an envelope function that decays to 0 for large $|x|$, $\frac{\sin\,(\Delta x)}{\Delta x}$.

4. The width of the envelope is $\delta x = \frac{1}{\Delta}$, since it can be written as $\frac{\sin\left(\frac{x}{1/\Delta}\right)}{\frac{x}{1/\Delta}}$. The width of $h(k)$ is $\delta k = 2\Delta$, so $\delta k = \frac{2}{\delta x}$, as in the case of a Gaussian wave packet.

Water-surface wave dispersion

1. Recall the dispersion relation is given by $\omega(k) = \sqrt{\frac{4kf(k)}{\rho}} = \sqrt{gk + \frac{\sigma k^3}{\rho}}$, where the last expression is the result of $f(k) = \frac{1}{4}\rho g + \frac{1}{4}\sigma k^2$, which we found in the first problem. As $\lambda \to \infty, k \to 0$, so for long wavelengths, $\frac{\sigma k^3}{\rho} \ll gk$, and the dispersion relation is approximately
$$\omega(k \to 0) \approx \sqrt{gk}.$$
2. For $\lambda \to 0, k \to \infty$, so for short wavelengths, $\frac{\sigma}{\rho}k^3 \gg gk$, and the dispersion relation is approximately
$$\omega(k \to \infty) \approx \sqrt{\frac{\sigma k^3}{\rho}}.$$
3. The speed of a single mode k is given by $v = \frac{\omega}{k}$. For the long wavelength regime (small k), we have $v \approx \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$. This increases with wavelength, so in this regime, the longer wavelength modes will travel faster. In the short wavelength regime, we have $v \approx \frac{\sqrt{\frac{\sigma}{\rho}k^3}}{k} = \sqrt{\frac{\sigma}{\rho}k} = \sqrt{\frac{2\pi g}{\rho\lambda}}$. This decreases with wavelength, so in this regime, the shorter wavelength modes will travel faster. In the top animation, we see the longer wavelengths travel faster, and in the lower animation, we see the opposite is true. Therefore, the top animation corresponds to longer wavelengths.