Homework 5

Due date: Thursday, December 1

1. For $q = 2$, the cubic term in the Potts model field theory vanishes and we have to include one further term when expanding the result of the Hubbard-Stratonovich transformation. The resulting field theory for this case, which describes the Ising model, has Hamiltonian

$$\mathcal{H} = \int_{\mathcal{V}} d^D\mathbf{r} \left(\frac{1}{2}|\nabla\Psi|^2 + \frac{r}{2}\Psi^2 - h\Psi + u\Psi^4 \right).$$

   Notice that the field has only $q-1 = 1$ component in this case. Repeat for this Hamiltonian the mean field analysis that was carried out in lecture for the case $1 < q < 2$. The results of your analysis should include the mean-field critical exponents for spontaneous magnetization (β) and susceptibility (γ), as well as the upper critical dimension D_c.

2. By closely mirroring the in-lecture calculation of $\langle \mathcal{H}'^2 \rangle_{\mathrm{conn}}$, that gave us $\delta c/\delta\lambda$ and $\delta r/\delta\lambda$, calculate $\langle \mathcal{H}'^3 \rangle_{\mathrm{conn}}$ and show that

$$\frac{\delta w}{\delta\lambda} = 36(q-3)w^3 \frac{\Omega_D K^D}{(K^2+r)^3}.$$

   The following are some checks on parts of the calculation:

   (a) There are only two Feynman diagrams to be considered. Draw both of them, not because they will help you do the calculation, but because there is no better way to impress your friends!

   One of the diagrams can be neglected because there is no way for the three internal momentum variables (associated with the traced-out fields) to all be in the momentum shell while having the external momenta (associated with the un-traced fields) be small.

   (b) The single Feynman diagram that contributes will have in it the factor

$$I(p,p') = \int^{>} \frac{d^Dk}{(2\pi)^D} \int^{>} \frac{d^Dk'}{(2\pi)^D} \int^{>} \frac{d^Dk''}{(2\pi)^D} \frac{(2\pi)^D\delta^D(p-k''+k)(2\pi)^D\delta^D(p'-k+k')}{(k^2+r)(k'^2+r)(k''^2+r)},$$

   where p, p' and $-p-p'$ are the three small external momenta. Since we are only interested in the renormalization of the low momentum limit of the cubic term, you only need to evaluate $I(0,0)$.

(c) The q-ology for the relevant Feynman diagram involves the sum (over repeated indices)

$$S_{iln} = Q_{ijk}Q_{klm}Q_{mjn}.$$

When you evaluate this sum you will find that, not surprisingly, it is proportional to the only permutation-invariant 3-index tensor at hand, Q_{iln}.

3. In terms of the rescaled parameters

$$\tilde{r} = r/K^2 \qquad \tilde{w} = w\sqrt{\Omega_D K^{D-6}}$$

the flow equations for $q = 1$ (percolation) take the form

$$\begin{aligned} \dot{\tilde{r}} &= 2\tilde{r} + 18\tilde{w}^2 - 30\tilde{r}\tilde{w}^2 \\ \dot{\tilde{w}} &= \frac{\epsilon}{2}\tilde{w} - 63\tilde{w}^3. \end{aligned}$$

(a) Find the fixed points of this system to lowest order in ϵ, both for $D > 6$ ($\epsilon < 0$) and $D < 6$ ($\epsilon > 0$). Do you find that the effects of fluctuations raise or lower the ordering temperature of the $q \to 1$ Potts model?

(b) Determine to lowest order in ϵ the eigenvectors and eigenvalues of the flow equations linearized about the fixed points.