Homework 5

Due date: Wednesday, March 6

Water-surface wave potential energy

Consider a water-surface height function (at time $t = 0$) having the form

$$s(x) = h\cos(kx)\ .$$

1. Calculate the change $\Delta U_g/\mathcal{A}$ in the gravitational potential energy (per area of surface) relative to the flat, $h = 0$ surface.
2. Calculate the increase $\Delta\mathcal{A}$ in the surface area of the water for the same surface height function. Do this in the limit of small h so the slope of the surface is always small and the following approximation can be used:

$$\begin{aligned}\text{arc-length} &= \int \sqrt{dx^2 + ds^2} \\ &= \int \sqrt{1 + \left(\frac{ds}{dx}\right)^2}\, dx \\ &\approx \int \left(1 + \frac{1}{2}\left(\frac{ds}{dx}\right)^2\right) dx\ .\end{aligned}$$

3. Calculate the corresponding increase in surface energy per area $\mathcal{A}$ of surface

$$\Delta U_\sigma/\mathcal{A} = \sigma\Delta\mathcal{A}/\mathcal{A}\ ,$$

where σ is the surface tension of water.

Motion of fluid elements in water-surface waves

In a **standing** water surface wave the fluid velocity (below the surface) has the form

$$\begin{aligned}v_x &= h\sin(\omega t)\sin(kx)e^{kz} \\ v_z &= -h\sin(\omega t)\cos(kx)e^{kz}\ .\end{aligned}$$

Using superposition we constructed in lecture the fluid velocity for a **running** water surface wave:

$$\begin{aligned}v_x &= h\left(\sin(\omega t)\sin(kx) + \cos(\omega t)\cos(kx)\right)e^{kz} \\ v_z &= h\left(-\sin(\omega t)\cos(kx) + \cos(\omega t)\sin(kx)\right)e^{kz}\ .\end{aligned}$$

1. What function $s(x,t)$ describes the height of the water surface (relative to the equilibrium surface at $z = 0$) for a standing wave? For a running wave?
2. Now consider a fluid element near (x, z). From the velocities above, of fluid elements near (x, z), obtain the position (p_x, p_z) of the fluid element as a function of time for both types of wave. Your position should be periodic in time. Choose the absolute position of the fluid element so its average position is (x, z).
3. You will find that for one type of wave the fluid elements execute linear motion, while for the other the fluid elements move in circles. Make a sketch showing how both types of motion vary with x and z.

Rectangular wave packets

Instead of forming a wave packet

$$s(x,0) = \int_{-\infty}^{\infty} h(k)\cos(kx)dk$$

using a Gaussian for the distribution $h(k)$, use a "rectangle distribution" with these properties

- It is symmetric about $k = k_0$ and has k_0 as its average.
- It has a width-independent normalization.
- It is constant over a finite range 2Δ in k and is zero outside that range.

1. Make a sketch of your rectangle distribution.
2. Do the integral to find $s(x, 0)$ without any outside help (the integrand is just the exponential function). While it's fine to use complex numbers, your final answer should be something explicitly real since $h(k)$ is real.
3. Express $s(x, 0)$ as the product of an infinite oscillatory function and an "envelope function" that decays to zero for large argument (of either sign).
4. Confirm that the width δk of $h(k)$ and the width δx of the "envelope" have the same kind of relationship they had for the Gaussian wave packet.

Water-surface wave dispersion

1. Write an approximation of the water-surface wave dispersion relation that is valid for long wavelengths.
2. Write an approximation of the water-surface wave dispersion relation that is valid for short wavelengths.

3. On the course webpage you will find two animations of dispersing water-surface waves. In one the scale is very large, showing the behavior of waves on the ocean. In the other the scale is small, more suitable for waves on the surface of an aquarium. Which is which? To get full credit on this problem you not only have to get the right answer, but explain (correctly) what features of the animation led you to your answer.