## Assignment 5

Due date: Tuesday, October 3

## Vector field decomposition I

Given the definitions in lecture of $\mathbf{v}_{L}$ and $\mathbf{v}_{T}$, for a general vector field $\mathbf{v}$ in three dimensions (no boundaries), show that $\nabla \times \mathbf{v}_{L}=0$ and $\nabla \cdot \mathbf{v}_{T}=0$.

## Vector field decomposition II

The figure below shows one period of the following periodic vector field in two dimensions:

$$
\mathbf{v}=\left(v_{x}, v_{y}\right)=\left(\cos x \sin ^{2} y,-\cos y \sin ^{2} x\right)
$$



Notice that this field has both non-vanishing divergence and non-vanishing circulation: it is neither transverse nor longitudinal.
In this problem you will use the discrete Fourier transform to numerically perform the decomposition $\mathbf{v}=\mathbf{v}_{\mathrm{T}}+\mathbf{v}_{\mathrm{L}}$ into transverse and longitudinal fields. If you do not have access to, or experience with, software for computing FFTs or rendering vector fields, team up with someone who does.

The first step is to choose the grid dimension $N$. The grid points x have integer coordinates that run from 0 to $N-1$. Next, compute the 2D FFTs of $v_{x}$ and $v_{y}$, sampled on this grid. We can write the relationship between the two representations of the vector fields as follows,

$$
\begin{equation*}
\mathbf{v}(\mathbf{x})=\frac{1}{(\sqrt{N})^{2}} \sum_{\mathbf{k}} e^{i 2 \pi(\mathbf{k} \cdot \mathbf{x}) / N} \tilde{\mathbf{v}}(\mathbf{k}) \tag{1}
\end{equation*}
$$

where the sum is over another 2D grid of points k with integer coordinates. This grid also has size $N \times N$ because the phase factor is unchanged when any multiple of $N$ is added to a component of $\mathbf{k}$. However, the choice of $\mathbf{k}$ will affect the decomposition into transverse and longitudinal. As an extreme case consider the k's $(1,0)$ and $(-1,0)$. Both of these correspond to the smallest possible spatial variation on the grid. However, the second of these is equivalent to ( $N-1,0$ ) - a very large spatial variation that would only be noticed if one could sample between the integer points x . To get the correct continuum limit from our grid samples we should always choose the k with the smallest magnitude. In practice this means each component of k runs as $0,1,2, \ldots,-2,-1$ instead of $0,1,2, \ldots, N-2, N-1$.
The longitudinal and transverse projections of $\tilde{\mathbf{v}}$ are defined by:

$$
\begin{aligned}
\tilde{\mathbf{v}}_{\mathrm{L}}(\mathbf{k}) & =(\tilde{\mathbf{v}}(\mathbf{k}) \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} \\
\tilde{\mathbf{v}}_{\mathrm{T}}(\mathbf{k}) & =\tilde{\mathbf{v}}(\mathbf{k})-(\tilde{\mathbf{v}}(\mathbf{k}) \cdot \hat{\mathbf{k}}) \hat{\mathbf{k}} .
\end{aligned}
$$

By replacing $\tilde{\mathbf{v}}$ in (1) with $\tilde{\mathbf{v}}_{\mathrm{L}}$, the resulting vector field will be a superposition of waves whose polarization is always parallel to the direction of propagation ( $\hat{\mathbf{k}}$ ); the perpendicular relationship holds when we use $\tilde{\mathbf{v}}_{\mathrm{T}}$ instead. The last step is therefore to undo the earlier FFTs after making these replacements, thus producing the vector fields $\mathbf{v}_{\mathrm{L}}$ and $\mathbf{v}_{\mathrm{T}}$. Make plots to visually confirm the curl-free and divergence-free properties.

Fields produced by a current sheet
The current density in an infinite sheet has the form

$$
\mathbf{J}(\mathbf{x}, t)=c \sigma \delta(z) \cos k x \cos \omega t \hat{\mathbf{y}}
$$

and the charge density $\rho$ vanishes everywhere at all times (the constant $\sigma$ has units of surface charge density).

1. Is charge conserved? Is the current density everywhere transverse?
2. Calculate $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ everywhere in space and time; consider separately the cases $\omega<c k$ and $\omega>c k$. Hint: first argue that

$$
\mathbf{A}(\mathbf{x}, t)=\operatorname{Re}\left(f(z) e^{-i \omega t}\right) \cos k x \hat{\mathbf{y}}
$$

is a valid expression for the vector potential in this problem.
3. Calculate the time-averaged power radiated by the sheet per unit area of the sheet.

