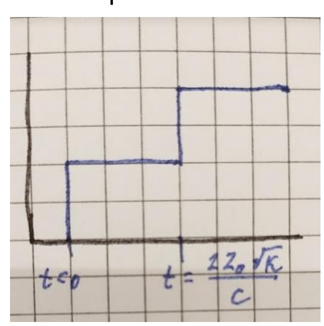
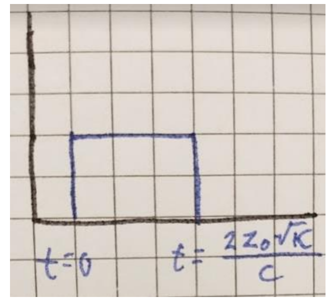
Co-ax boundary conditions

1. We know the surface charge density at the inner conductor, with radius r, is $\sigma(z,t) = \frac{\kappa\epsilon_0}{a} f(z,t)$. We also know f satisfies the wave equation with velocity $\frac{c}{\sqrt{\kappa}}$, which means $\sigma(z,t)$ also satisfies the wave equation with velocity $\frac{c}{\sqrt{\kappa}}$. The most general solution to the wave equation, as we have seen, is $\tilde{f}(z - vt) + \tilde{g}(z + vt)$ (I have used $\tilde{f}$ and $\tilde{g}$ to avoid confusion with the f and g the appear in the expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$), so the most general form of σ is
$$\sigma(z,t) = \tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) + \tilde{g}\left(z + \frac{c}{\sqrt{\kappa}}t\right).$$
2. In the previous homework we saw the equation implying conservation of charge is $\frac{\partial j}{\partial z} = -\frac{\partial \sigma}{\partial t}$, which means $\frac{\partial j}{\partial z} = \frac{c}{\sqrt{\kappa}}\left(\tilde{f}'\left(z - \frac{c}{\sqrt{\kappa}}t\right) - \tilde{g}'\left(z + \frac{c}{\sqrt{\kappa}}t\right)\right) = \frac{c}{\sqrt{\kappa}}\frac{\partial}{\partial z}\left(\tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) - \tilde{g}\left(z + \frac{c}{\sqrt{\kappa}}t\right)\right)$, which means
$$j(z,t) = \frac{c}{\sqrt{\kappa}}\left(\tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) - \tilde{g}\left(z + \frac{c}{\sqrt{\kappa}}t\right)\right)$$
3. If the cable is open-ended at $z = 0$, the current at that point will be 0, so $j(0,t) = \frac{c}{\sqrt{\kappa}}\left(\tilde{f}\left(-\frac{c}{\sqrt{\kappa}}t\right) - \tilde{g}\left(\frac{c}{\sqrt{\kappa}}t\right)\right) = 0$, which means $\tilde{f}(-y) = \tilde{g}(y)$. In this case we have
$$\sigma(z,t) = \tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) + \tilde{f}\left(-z - \frac{c}{\sqrt{\kappa}}t\right)$$
$$j(z,t) = \frac{c}{\sqrt{\kappa}}\left(\tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) - \tilde{f}\left(-z - \frac{c}{\sqrt{\kappa}}\right)\right)$$
4. In the case where the cable is shorted at $z = 0$, we have that the potential there must be 0, meaning the electric field is 0, and therefore $\sigma = 0$, since σ and E are both proportional to f. Therefore $\sigma(0,t) = \tilde{f}\left(-\frac{c}{\sqrt{\kappa}}t\right) + \tilde{g}\left(\frac{c}{\sqrt{\kappa}}t\right) = 0$, which means $-\tilde{f}(-y) = \tilde{g}(y)$. In this case, we have
$$\sigma(z,t) = \tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) - \tilde{f}\left(-z - \frac{c}{\sqrt{\kappa}}t\right)$$
$$j(z,t) = \frac{c}{\sqrt{\kappa}}\left(\tilde{f}\left(z - \frac{c}{\sqrt{\kappa}}t\right) + \tilde{g}\left(-z - \frac{c}{\sqrt{\kappa}}t\right)\right).$$
5. The reflected pulse is $\tilde{g}$, which in this case will be positive since $\tilde{g}(y) = \tilde{f}(-y)$, and $\tilde{f}$ is positive. If the oscilloscope is connected at $z = z_0$, the time from when the pulse is sent to when the reflected pulse is seen is $\frac{2z_0\sqrt{\kappa}}{c}$, since the pulse must get to $z = 0$ from $z = z_0$ and then the reflected has to get to $z = z_0$ from $z = 0$ with speed $\frac{c}{\sqrt{\kappa}}$. Thus at $t = \frac{2z_0\sqrt{\kappa}}{c}$ the two pulses will add together. An example of a possible trace sketch is shown.



6. The reflected pulse will be $\tilde{g}$ negative since $\tilde{g}(y) = -\tilde{f}(-y)$, and $\tilde{f}$ is positive. At $t = \frac{2z_0\sqrt{\kappa}}{c}$ the reflected pulse will cancel out the emitted pulse. An example of a possible trace sketch is shown.



Hanging chain normal modes

1. Letting $y(z,t) = \cos(\omega t)\, f_\omega(z)$, we find $f_\omega(z)\frac{d^2\cos(\omega t)}{dt^2} = g\cos(\omega t)\frac{d}{dz}\left(z\frac{df_\omega(z)}{dz}\right)$. The left side becomes $-\omega^2 f_\omega(z)\cos(\omega t)$, while the right side becomes $g\cos(\omega t)\ \left(\frac{df_\omega(z)}{dz} + z\frac{d^2 f_\omega(z)}{dz^2}\right)$. Dividing out the common factor of $\cos(\omega t)$ and rearranging, we have

$$z\frac{d^2 f_\omega(z)}{dz^2} + \frac{df_\omega(z)}{dz} + \frac{\omega^2}{g} f_\omega(z) = 0.$$

2. Let $u = c_\omega\sqrt{z}$. Then, $z = \frac{u^2}{c_\omega^2}$, $\frac{d}{dz} = \frac{du}{dz}\frac{d}{du} = \frac{1}{2}\frac{c_\omega}{\sqrt{z}}\frac{d}{du} = \frac{1}{2}\frac{c_\omega^2}{u}\frac{d}{du}$, and $\frac{d^2}{dz^2} = \frac{d}{dz}\left(\frac{d}{dz}\right) = \frac{1}{2}\frac{c_\omega^2}{u}\frac{d}{du}\left(\frac{1}{2}\frac{c_\omega^2}{u}\frac{d}{du}\right) = \frac{1}{4}\frac{c_\omega^4}{u}\frac{d}{du}\left(\frac{1}{u}\frac{d}{du}\right) = \frac{c_\omega^4}{4}\left(\frac{1}{u^2}\frac{d^2}{du^2} - \frac{1}{u^3}\frac{d}{du}\right)$. With this, we have
$$z\frac{d^2 f_\omega(z)}{dz^2} + \frac{df_\omega(z)}{dz} + \frac{\omega^2}{g} f_\omega(z) = 0 \rightarrow \frac{u^2}{c_\omega^2}\frac{c_\omega^4}{4}\left(\frac{1}{u^2}\frac{d^2 f_\omega}{du^2} - \frac{1}{u^3}\frac{df_\omega}{du}\right) + \frac{1}{2}\frac{c_\omega^2}{u}\frac{df_\omega}{du} + \frac{\omega^2}{g} f_\omega = 0$$
Simplifying the second equation, we get $\frac{c_\omega^2}{4}\frac{d^2 f_\omega}{du^2} + \frac{c_\omega^2}{4}\frac{1}{u}\frac{df_\omega}{du} + \frac{\omega^2}{g} f_\omega = 0$. Multiplying across by $\frac{4u}{c_\omega^2}$, we finally have
$$u\frac{d^2 f_\omega}{du^2} + \frac{df_\omega}{du} + \frac{4\omega^2}{gc_\omega^2}\, u\, f_\omega = 0,$$
Which is exactly the defining equation of $J_0(u)$ provided $\frac{4\omega^2}{gc_\omega^2} = 1$, which implies $c_\omega = \frac{2\omega}{\sqrt{g}}$, and therefore $f_\omega(z) = J_0(u) = J_0\left(2\omega\sqrt{\frac{z}{g}}\right)$.

3. We know $f_\omega(L) = 0$, because the chain is fixed at $z = L$. That means $J_0\left(2\omega\sqrt{\frac{L}{g}}\right) = 0$, which implies $2\omega\sqrt{\frac{L}{g}}$ is a zero of J_0. Substituting $\omega = \frac{2\pi}{T}$, we have $\frac{4\pi}{T}\sqrt{\frac{L}{g}}$ is a zero of J_0. Therefore the periods of the first three normal modes are
$$\frac{4\pi}{2.40483}\sqrt{\frac{L}{g}}, \frac{4\pi}{5.52008}\sqrt{\frac{L}{g}}, \frac{4\pi}{8.65373}\sqrt{\frac{L}{g}}.$$

Circuit model for myelinated nerve fibers

1. In homework 2 we saw the capacitance for a parallel plate capacitor is given by $C = \kappa\epsilon_0\frac{A}{d}$, where A is the area of the plates and d the thickness between the plates. In this case, the

"plates" are actually cylinders with radius r (we can ignore the difference in radius between the cylinders), and width w. The area is then $2\pi rw$. The capacitance is then

$$C = \frac{2\pi\kappa\epsilon_0 rw}{d}.$$

With $r = 5\mu m$, $w = 1\mu m$, $d = 6\ nm$, and $\kappa = 7$, this gives

$$C = 14\pi\epsilon_0 \frac{5}{6} \times 10^{-3}\mathrm{m} \ = 3.2 \times 10^{-13}\ \mathrm{F} = 0.32\ \mathrm{pF}$$

We'll keep an extra significant figure to prevent rounding error in the next parts.

2. The resistance R is related to the resistivity by $R = \frac{\rho L}{A}$, where A is the cross-sectional area, and L is the length of the resistor. The cross-sectional area is πr^2, so

$$R = \frac{\rho L}{\pi r^2}.$$

With $r = 5\mu m$, $L = 1\ mm$, and $\rho = 1\ \Omega m$, we have

$$R = \frac{10^9}{25\pi}\ \Omega = 1.3 \times 10^7\ \Omega = 13\ \mathrm{M\Omega}.$$

3. The voltage at the nth capacitor is $\frac{q(n,t)}{C}$, and the voltage drop across the nth resistor is $V = i(n,t)R$. Therefore, $\frac{q(n,t)}{C} - i(n,t)R = \frac{q(n+1,t)}{C}$. Solving for $i(n,t)$, we get $i(n,t) = \frac{1}{RC}\big(q(n,t) - q(n+1,t)\big)$. Taking the derivative with respect to time, we find $\frac{\partial i(n,t)}{\partial t} = \frac{1}{RC}\left(\frac{\partial q(n,t)}{\partial t} - \frac{\partial q(n+1,t)}{\partial t}\right)$, but $\frac{\partial q(n,t)}{\partial t}$ is the rate of change of the current in the nth capacitor, which is the difference between $i(n-1,t)$, the current flowing into the capacitor from the previous segment, and $i(n,t)$, the current flowing out of it into the next segment. Therefore, $\frac{\partial q(n,t)}{\partial t} = \ i(n-1,t) - i(n,t)$. Substituting this in, we have

$$\frac{\partial i(n,t)}{\partial t} = \frac{1}{RC}\ \big(i(n-1,t) - 2i(n,t) - i(n+1,t)\big).$$

In the continuous limit, the part in parenthesis as we've seen before becomes $\Delta x^2 \frac{\partial^2 i(t)}{\partial x^2}$, with $\Delta x = L$, the length of one cell, and the equation becomes

$$\frac{\partial i(\ t)}{\partial t} = \frac{\Delta x^2}{RC}\frac{\partial^2 i}{\partial x^2}.$$

The time constant RC is

$$RC = 1.3 \times 10^7 \times 3.2 \times 10^{-13}\mathrm{s} = 4 \times 10^{-6}\ \mathrm{s} = 4\ \mathrm{\mu s}.$$

This is the time for a single cell to respond. In the example of a pianist, the distance between the fingers and the brain is around $1\ \mathrm{m}$, or about 1000 cells. The total response time is about $4\ \mathrm{ms}$. 20 notes per second gives a time of $0.05\ \mathrm{s} = 50\ \mathrm{ms} = 5 \times 10^4\ \mathrm{\mu s}$, or about 10 times the time calculated above, but as pointed out, this is not really the unconscious reaction time.

4. The equation derived in part 3 does not have time reversal symmetry, since $\frac{\partial i}{\partial t} = -\frac{\partial i}{\partial t'}$, where $t' = -t$.