Homework 4

Due date: Tuesday, November 1

1. For the q-state Potts model in the mean field approximation: Find the transition temperature, nature of the transition, and the critical exponents for magnetization and susceptibility if the transition is continuous. The Potts model Hamiltonian is

$$H = -\epsilon \sum_{(ij)\in\mathcal{E}} (q\delta(s_i, s_j) - 1) - h \sum_{i\in\mathcal{V}} (q\delta(s_i, 1) - 1)\,,$$

where $\mathcal{E}$ and $\mathcal{V}$ are respectively the edges and vertices of a hypercubic lattice graph in D-dimensions (each vertex has degree $2D$). In the mean field approximation, the expression

$$\beta F = \beta\langle H\rangle - S$$

for the free energy F is minimized over the probability space where all spins have equal and independent distributions. This probability distribution has only one parameter, the excess probability m that a spin takes value $s = 1$:

$$p_1 = 1/q + m.$$

Here is the outline of your mean field calculation:

   (a) Calculate $\langle H\rangle$.

   (b) Calculate S.

   (c) Find the value m^* that minimizes F in the limit $h \to 0^+$. You will obtain qualitatively different results in three cases: $q > 2$, $q = 2$, and $2 > q > 1$.

   (d) Find the transition temperature, where m^* first becomes nonzero. In the percolation problem ($q \to 1$) the transition temperature is related to the critical bond probability by $p_c = 1 - e^{-\beta_c \epsilon}$. What is the mean field estimate of p_c?

   (e) (e) Obtain the mean field estimates of the critical exponents β and γ defined by

$$\begin{aligned} m^*|_{h=0} &\propto (T_c - T)^\beta \\ \left.\frac{\partial m^*}{\partial h}\right|_{h=0} = \chi &\propto (T_c - T)^{-\gamma} \end{aligned}$$

   for $T \to T_c^-$ when the transition is continuous. Compare with the percolation exponents for f (fraction in finite clusters) and s ("size" of clusters).

2. As the first step in transforming the Potts model into a field theory we rewrite the Hamiltonian in terms of vector variables $\mathbf{v}(\mathbf{r})$ on the sites $\mathbf{r}$ of the lattice,

$$H = -\epsilon \sum_{(\mathbf{r},\mathbf{r}')\in\mathcal{E}} \mathbf{v}(\mathbf{r})\cdot\mathbf{v}(\mathbf{r}') - h\sum_{\mathbf{r}\in\mathcal{V}} \mathbf{e}^1\cdot\mathbf{v}(\mathbf{r}),$$

and the vectors take discrete values in the set $\{\mathbf{e}^1,\ldots,\mathbf{e}^q\}$. These "state-vectors" live in a $q-1$ dimensional Euclidean space and form the vertices of a regular q-simplex. In order that the vector form of the Potts Hamiltonian agrees with the original form we require that

$$\begin{aligned} \mathbf{e}^\alpha\cdot\mathbf{e}^\alpha &= q-1, \qquad \alpha = 1,\ldots,q \\ \mathbf{e}^\alpha\cdot\mathbf{e}^\beta &= -1, \qquad \alpha\neq\beta, \end{aligned}$$

which together imply

$$\sum_{\alpha=1}^{q}\mathbf{e}^\alpha = 0.$$

Show that these properties follow from the embedding of the q state-vectors in $q-1$ dimensions given by the rows of the following $q\times(q-1)$ matrix,

$$\begin{bmatrix} a & -b & -b & \cdots & -b \\ -b & a & -b & \cdots & -b \\ -b & -b & a & \cdots & -b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b & -b & -b & \cdots & a \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix}$$

for a particular choice of the numbers a and b. Also show that

$$\sum_{\alpha=1}^{q} e_i^\alpha e_j^\alpha = q\delta_{ij}.$$

3. The adjacent lattice site (edge) term of $-\beta H$ for the Potts model Hamiltonian (problem 2) may be written abstractly as $V^{\mathrm{T}}KV$, where V is a vector of $(q-1)\mathcal{V}$ components (corresponding to the $q-1$ vector components of the $\mathbf{v}(\mathbf{r})$'s at all $\mathcal{V}$ lattice sites $\mathbf{r}$ in the system). The Hubbard-Stratonovich transformation converts the trace over the discrete variables V into a trace over continuous fields Ψ in one-to-one correspondence with the components of V. An integral part of the transformation is the evaluation of the associated coupling of the fields, $\Psi^{\mathrm{T}}K^{-1}\Psi$. Show that in the

limit of slowly varying fields, so that the sum over sites may be approximated by an integral,

$$\Psi^{\mathrm{T}} K^{-1} \Psi \approx (\beta\epsilon D)^{-1} \int d^D\mathbf{r} \sum_{i=1}^{q-1} \left(\Psi_i^2 + \frac{1}{2D} |\nabla \Psi_i|^2 \right).$$

[Hint: The eigenvectors of the matrix K are plane waves in $\mathbf{r}$ with constant polarization in the $(q-1)$-space of the Potts model vectors $\mathbf{v}$. The action of K^{-1} on these eigenvectors is just to multiply the eigenvector by the inverse of the eigenvalue. Express the most general Ψ in terms of these eigenvectors and thereby obtain a formula for $\Psi^{\mathrm{T}} K^{-1} \Psi$.]