Lorentz invariance of the wave equation

We need to compute the partial derivatives in terms of partial derivatives with respect to the primed coordinates.

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \gamma_u \frac{\partial}{\partial t'} - u\gamma_u \frac{\partial}{\partial x'}$$
$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = -\frac{\gamma_u u}{v^2}\frac{\partial}{\partial t'} + \gamma_u \frac{\partial}{\partial x'}$$

The second derivatives are then

$$\frac{\partial^2}{\partial t^2} = \left(\gamma_u \frac{\partial}{\partial t'} - u\gamma_u \frac{\partial}{\partial x'}\right)\left(\gamma_u \frac{\partial}{\partial t'} - u\gamma_u \frac{\partial}{\partial x'}\right) = \gamma_u^2 \left(\frac{\partial^2}{\partial t'^2} - u\frac{\partial^2}{\partial t'\partial x'} - u\frac{\partial^2}{\partial x'\partial t'} + u^2\frac{\partial^2}{\partial x'^2}\right)$$

$$\frac{\partial^2}{\partial x^2} = \left(-\frac{\gamma_u u}{v^2}\frac{\partial}{\partial t'} + \gamma_u \frac{\partial}{\partial x'}\right)\left(-\frac{\gamma_u u}{v^2}\frac{\partial}{\partial t'} + \gamma_u \frac{\partial}{\partial x'}\right) = \gamma_u^2 \left(\frac{u^2}{v^4}\frac{\partial^2}{\partial t'^2} - \frac{u}{v^2}\frac{\partial^2}{\partial t'\partial x'} - \frac{u}{v^2}\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial^2}{\partial x'^2}\right)$$

We can now write $\frac{\partial^2\Psi}{\partial t^2} = v^2\frac{\partial^2\Psi}{\partial x^2}$ as follows

$$\gamma_u^2\left(\frac{\partial^2\Psi}{\partial t'^2} - u\frac{\partial^2\Psi}{\partial t'\partial x'} - u\frac{\partial^2\Psi}{\partial x'\partial t'} + u^2\frac{\partial^2\Psi}{\partial x'^2}\right) = \gamma_u^2 v^2\left(\frac{u^2}{v^4}\frac{\partial^2\Psi}{\partial t'^2} - \frac{u}{v^2}\frac{\partial^2\Psi}{\partial t'\partial x'} - \frac{u}{v^2}\frac{\partial^2\Psi}{\partial x'\partial t'} + \frac{\partial^2\Psi}{\partial x'^2}\right)$$

Dividing both sides by γ_u^2 and distributing the v^2 on the right, we have

$$\frac{\partial^2\Psi}{\partial t'^2} - u\frac{\partial^2\Psi}{\partial t'\partial x'} - u\frac{\partial^2\Psi}{\partial x'\partial t'} + u^2\frac{\partial^2\Psi}{\partial x'^2} = \frac{u^2}{v^2}\frac{\partial^2\Psi}{\partial t'^2} - u\frac{\partial^2\Psi}{\partial t'\partial x'} - u\frac{\partial^2\Psi}{\partial x'\partial t'} + v^2\frac{\partial^2\Psi}{\partial x'^2}$$

Cancelling out the common mixed derivatives terms and grouping derivatives with respect to t' on the left and derivatives with resect to x' on the right, we have

$$\left(1 - \frac{u^2}{v^2}\right)\frac{\partial^2\Psi}{\partial t'^2} = v^2\left(1 - \frac{u^2}{v^2}\right)\frac{\partial^2\Psi}{\partial x'^2},$$

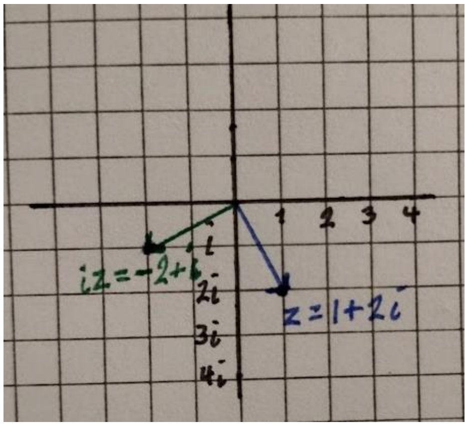
Where on the right we have factored out a v^2. Dividing out the common factor $\left(1 - \frac{u^2}{v^2}\right)$, we have

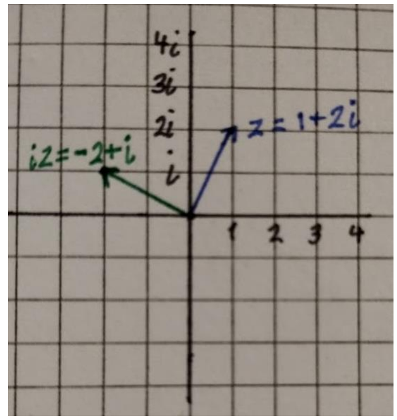
$$\frac{\partial^2\Psi}{\partial t'^2} = v^2\frac{\partial^2\Psi}{\partial x'^2}.$$

Symmetries of the Schrödinger equation

1. Using the chain rule, we have $\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\frac{\partial}{\partial t'}$. This transforms the Schrödinger equation into
$$-i\hbar\frac{\partial\Psi}{\partial t'} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2},$$
Which is not the same as the original equation, due to the negative sign on the left.
2. Squaring $-i$ we get $(-i)^2 = (-1)^2 i^2 = i^2 = -1$

3. In both cases multiplying by i corresponds to rotating the vector by 90 degrees.





4. Conjugation transforms $i\hbar$ to $-i\hbar$, and time reversal transforms $\frac{\partial}{\partial t}$ to $-\frac{\partial}{\partial t'}$ as we saw in part 1. Together, we then have $i\hbar\frac{\partial\Psi(\mathrm{x,t})}{\partial t} \to (-i\hbar)\left(-\frac{\partial\Psi'(x,t')}{\partial t'}\right) = i\hbar\frac{\partial\Psi'}{\partial t'}$. The Schrödinger equation then becomes

$$i\hbar\frac{\partial\Psi'}{\partial t'} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi'}{\partial x^2},$$

Which is the same as the untransformed Schrödinger equation.

5. We lose amplitude translation symmetry, since writing the equation in terms of $\Psi' = \Psi - \Psi_0$ gives

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi$$

$$i\hbar\frac{\partial(\Psi' + \Psi_0)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2(\Psi' + \Psi_0)}{\partial x^2} + U(x)(\Psi' + \Psi_0)$$

$$i\hbar\frac{\partial\Psi'}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi'}{\partial x^2} + U(x)\Psi' + U(x)\Psi_0$$

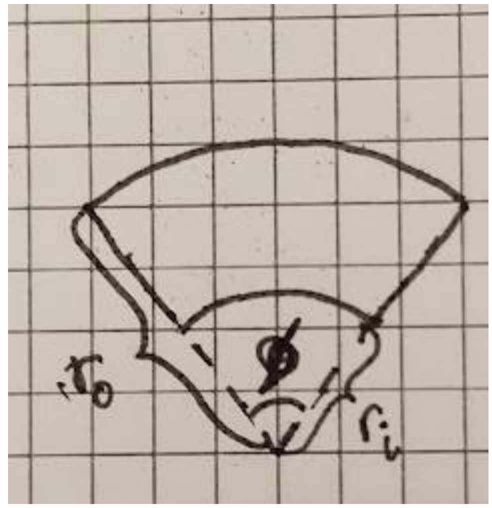
Which has an extra term $U(x)\Psi_0$ compared to the original Schrödinger equation. Additionally, we lose translation symmetry. $\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial^2\Psi}{\partial x'^2}$, where $x' = x - x_0$, so the Schrödinger equation now becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial {x'}^2} + U(x' + x_0)\Psi$$

Which is not the same as the original Schrödinger equation.

Propagating electric and magnetic fields in a co-ax

1. The divergence theorem states that $\int_V \nabla \cdot \boldsymbol{F}\, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{A}$, where V is a given volume and S is the surface area enclosing that volume, and $\boldsymbol{F}$ is a vector field, which in our case is $\boldsymbol{E}$ or $\boldsymbol{B}$. For our volume, we pick a wedge-shaped volume such that the inner surface is section of a cylinder of radius r_i and subtending an angle ϕ, and the outer surface is a section of a cylinder of radius r_o and subtending the angle ϕ. The figure on the right shows a cross section of this volume. Let the length of the volume be l. The electric field, being in the radial direction, runs along the sides and ends of this volume. Therefore, $\boldsymbol{E} \cdot d\boldsymbol{A} = 0$ for every area element on these surfaces. This means $\int_V \nabla \cdot \boldsymbol{E}\, dV = \int_{S_i} \boldsymbol{E} \cdot d\boldsymbol{A} + \int_{S_o} \boldsymbol{E} \cdot d\boldsymbol{A}$, where S_i and S_o are the inner and outer surfaces of the volume. Substituting in the expression for E, we have



$$\int_V \nabla \cdot \boldsymbol{E}\, dV = \int_{S_i} \frac{f(z,t)}{r_i} \hat{\boldsymbol{r}} \cdot d\boldsymbol{A} + \int_{S_o} \frac{f(z,t)}{r_o} \hat{\boldsymbol{r}} \cdot d\boldsymbol{A}$$

In cylindrical coordinates, the area elements on these surfaces are $r_i\, d\theta dz$ and $r_o\, d\theta dz$, and the direction of the surface vector is outward from the volume, which means it is $-\hat{\boldsymbol{r}}$ for the inner surface and $\hat{\boldsymbol{r}}$ for the outer surface. This then gives

$$\int_V \nabla \cdot \boldsymbol{E}\, dV = -\int_0^l \int_0^\phi \frac{f(z,t)}{r_i} r_i d\theta dz + \int_0^l \int_0^\phi \frac{f(z,t)}{r_o} r_o d\theta dz$$

$$= -\phi \int_0^l f(z,t) dz + \phi \int_0^l f(z,t) dz = 0$$

In summary, we have, $\int_V \nabla \cdot \boldsymbol{E}\, dV = 0$, which implies $\nabla \cdot \boldsymbol{E} = 0$.

For the magnetic field, since it is in the $\hat{\boldsymbol{\phi}}$ direction, it runs along the inner and outer surface, so $\boldsymbol{B} \cdot d\boldsymbol{A} = 0$ for every area element on those surfaces. It also runs along the ends of the volume, so the same applies there. This means $\int_V \nabla \cdot \boldsymbol{B}\, dV = \int_{S_1} \boldsymbol{B} \cdot d\boldsymbol{A} + \int_{S_2} \boldsymbol{B} \cdot d\boldsymbol{A}$ where S_1 and S_2 are the sides of the volume. Substituting in the expression for $\boldsymbol{B}$, we have

$$\int_V \nabla \cdot \boldsymbol{B}\, dV = \int_{S_1} \frac{g(z,t)}{r} \hat{\boldsymbol{\phi}} \cdot d\boldsymbol{A} + \int_{S_2} \frac{g(z,t)}{r} \hat{\boldsymbol{\phi}} \cdot d\boldsymbol{A}$$

In cylindrical coordinates, the area elements along the side surfaces have magnitude $dr\, dz$, and the direction of the surface vector is outward from the volume, so for one side it is $\hat{\boldsymbol{\phi}}$ and for the other side it is $-\hat{\boldsymbol{\phi}}$. With this, we have

$$\int_V \nabla \cdot \boldsymbol{B}\, dV = \int_{r_1}^{r_2} \int_0^l \frac{g(z,t)}{r} dr dz - \int_{r_1}^{r_2} \int_0^l \frac{g(z,t)}{r} dr dz = 0.$$

In summary, we have, $\int_V \nabla \cdot \boldsymbol{B}\, dV = 0$, which implies $\nabla \cdot \boldsymbol{B} = 0$.

2. First we compute $\nabla \times \boldsymbol{E}$. $\boldsymbol{E}$ only has an r component, which means $E_\phi = E_z = 0$. With this, the curl expression greatly simplifies to
$$\nabla \times \boldsymbol{E} = \frac{\partial E_r}{\partial z}\hat{\phi} - \frac{1}{r}\frac{\partial E_r}{\partial \phi}\hat{z}.$$
The second term also is zero, since $\boldsymbol{E}$ does not depend on ϕ. Substituting in $E_r = \frac{f(z,t)}{r}$, we have
$$\nabla \times \boldsymbol{E} = \frac{1}{r}\frac{\partial f(z,t)}{\partial z}\ \hat{\phi}.$$
For the case of $\boldsymbol{B}$, $B_r = B_z = 0$. In this case, the curl expression becomes
$$\nabla \times \boldsymbol{B} = -\frac{\partial B_\phi}{\partial z}\hat{r} + \frac{1}{r}\frac{\partial\left(rB_\phi\right)}{\partial r}\hat{z}.$$
After substituting in $B_\phi = \frac{g(z,t)}{r}$ we find that $rB_\phi = g(z,t)$, which is independent of r, so the second term is also zero. Consequently,
$$\nabla \times \boldsymbol{B} = -\frac{1}{r}\frac{\partial g(z,t)}{\partial z}\hat{r}.$$

3. The two curl equations are $\nabla \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$ and $\nabla \times \boldsymbol{B} = \frac{\kappa}{c^2}\dot{\boldsymbol{E}}$. Substituting in the expressions for $\boldsymbol{E}, \boldsymbol{B}$, and the curls from part 3, we have
$$\frac{1}{r}\frac{\partial f(z,t)}{\partial z}\ \hat{\phi} = -\frac{1}{r}\frac{\partial g(z,t)}{\partial t}\hat{\phi}, \frac{1}{r}\frac{\partial g(z,t)}{\partial z}\hat{r} = -\frac{\kappa}{c^2}\frac{1}{r}\frac{\partial f(z,t)}{\partial t}\hat{r}$$
Which can be simplified to
$$\frac{\partial f(z,t)}{\partial z} = -\frac{\partial g(z,t)}{\partial t}, \frac{\partial g(z,t)}{\partial z} = -\frac{\kappa}{c^2}\frac{\partial f(z,t)}{\partial t}.$$
Taking the derivative of the first equation with respect to z, we have
$$\frac{\partial^2 f(z,t)}{\partial z^2} = -\frac{\partial^2 g(z,t)}{\partial z \partial t}$$
And taking the derivative of the second equation with respect to t we have
$$\frac{\partial^2 g(z,t)}{\partial t \partial z} = -\frac{\kappa}{c^2}\frac{\partial^2 f(z,t)}{\partial t^2}.$$
Using the fact that, as far as us physicists are concerned, $\frac{\partial^2 g(z,t)}{\partial t \partial z} = \frac{\partial^2 g(z,t)}{\partial z \partial t}$, we can substitute the second equation into the first equation, and find
$$\frac{\partial^2 f(z,t)}{\partial z^2} = \frac{\kappa}{c^2}\frac{\partial^2 f(z,t)}{\partial t^2}.$$
This is the wave equation for waves propagating along z with speed $\frac{c}{\sqrt{\kappa}}$. We could have instead chosen to substitute in for f and found a second order partial differential equation for g, and we would have found the same wave equation. Since E, B are proportional to f, g, we have that E and B are waves propagating along z.
4. Inside the inner conductor, since the charge is a surface charge, a Gaussian surface will enclose no charge, and by Gauss's law the electric field will then be zero. Similarly, an Amperian loop inside the conductor will enclose no current, since the current is at the surface, and therefore the magnetic field will be zero. Inside the outer conductor, a Gaussian surface will enclose the charge from the inner conductor, as well as the surface charge from the inner surface of the outer conductor. These two charges are equal and

opposite, so the net enclosed charge will be zero, and so will the electric field. In the same way, an Amperian loop will enclose no net current, since the current of the inner conductor will cancel out the current at the inner surface of the outer conductor, and therefore the magnetic field will be zero. For a surface charge, we have $\int_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{1}{\epsilon_0}\int_S \sigma\, dA$, which for our case means $\int_0^{2\pi}\int_0^z \frac{f(z,t)}{a} adz\, d\phi = \frac{1}{\kappa\epsilon_0}\int_0^{2\pi}\int_0^z \sigma_a(z,t) adzd\phi$, where σ_a is the surface charge density at the surface with $r = a$.

Simplifying, we have $2\pi \int_0^z f(z,t)dz = \frac{2\pi a}{\kappa\epsilon_0}\int_0^z \sigma_a(z,t)dz$, which implies

$$\sigma_a(z,t) = \frac{\kappa\epsilon_0}{a} f(z,t).$$

For the magnetic field we have $\int_l \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_l jdl$, which in our case means $\int_0^{2\pi} \frac{g(z,t)}{a} ad\phi = \mu_0 \int_0^{2\pi} j_a(z,t) ad\phi$, where j_a is the surface current at the surface with $r = a$. Simplifying we get $2\pi g(z,t) = \ \mu_0 2\pi a j_a(z,t)$, which means

$$j_a(z,t) = \frac{g(z,t)}{\mu_0 a}.$$

Repeating the same steps with $r = b$ we find

$$\sigma_b(z,t) = \frac{\kappa\epsilon_0}{b} f(z,t), \qquad j_b(z,t) = \frac{g(z,t)}{\mu_0 b}.$$

We note that up to factors of $\kappa\epsilon_0$ and $\frac{1}{\mu_0}$ respectively, the charge density and current density are simply the values of the electric and magnetic fields at the conductors' surfaces, and therefore amount to the boundary conditions on the electromagnetic fields.

The amount of current flowing out of a strip of Δz is given by the total amount of current flowing out on one side, minus the current flowing in on the other side. That is, the total current flowing out is $j(z+\Delta z,t) - j(z,t) = \Delta z \frac{j(z+\Delta z,t) - j(z,t)}{\Delta z} \approx \Delta z \frac{\partial j}{\partial z} = \frac{\Delta z}{\mu_0 r_0}\frac{\partial g(z,t)}{\partial z}$, where r_0 can be either a or b. In part 3 we say $\frac{\partial g(z,t)}{\partial z} = -\frac{\kappa}{c^2}\frac{\partial f(z,t)}{\partial t}$, so $\frac{\Delta z}{\mu_0 r_0}\frac{\partial g(z,t)}{\partial z} = -\frac{\Delta z\kappa}{\mu_0 c^2}\frac{\partial f(z,t)/r_0}{\partial t}$. But we know $\sigma(z,t) = \kappa\epsilon_0 \frac{f(z,t)}{r_o}$, so $-\frac{\Delta z}{\mu_0 c^2}\frac{\partial}{\partial t}\frac{f(z,t)}{r_0} = -\frac{\Delta z\kappa}{\kappa\mu_0\epsilon_0 c^2}\frac{\partial\sigma(z,t)}{\partial t}$. Recalling $\frac{1}{\epsilon_0\mu_0} = c^2$, we have $\Delta z\frac{\partial j}{\partial z} = -\Delta z \frac{\partial\sigma(z,t)}{\partial t}, \frac{\partial j}{\partial z} = -\frac{\partial\sigma}{\partial t}$. That is, the current flowing out of the strip is equal to the rate at which the charge decreases, so charge is conserved.

5. Since $\sigma(z,t)$ is a pulse moving in the positive z direction, we have $\sigma(z,t) = h\left(z - \frac{c}{\sqrt{\kappa}}t\right)$, where $h(z)$ is some pulse-like function. Since $\sigma(z,t) = \kappa\epsilon_0 \frac{f(z,t)}{r_o}$, we have $f(z,t) = \frac{r_0}{\kappa\epsilon_0}\sigma(z,t) = \frac{r_0}{\kappa\epsilon_0} h\left(z - \frac{c}{\sqrt{\kappa}}t\right)$. This means the field is $E = \frac{f(z,t)}{r_0} = \frac{1}{\kappa\epsilon_0}\ h\left(z - \frac{c}{\sqrt{\kappa}}t\right)$, We also know $\frac{\partial g(z,t)}{\partial z} = -\frac{\kappa}{c^2}\frac{\partial f(z,t)}{\partial t}$, so $\frac{\partial g(z,t)}{\partial z} = -\frac{r_0}{c^2\epsilon_0}\frac{\partial h\left(z - \frac{c}{\sqrt{\kappa}}t\right)}{\partial t} = \frac{r_0}{c\sqrt{\kappa}\epsilon_0} h'\left(z - \frac{c}{\sqrt{\kappa}}t\right) = \frac{r_0}{c\sqrt{\kappa}\epsilon_0}\frac{\partial h\left(z - \frac{c}{\sqrt{\kappa}}t\right)}{\partial z}$,

which means $g(z,t) = \frac{r_0}{c\sqrt{\kappa}\epsilon_0} h\left(z - \frac{c}{\sqrt{\kappa}} t\right)$, and $B = \frac{g(z,t)}{r_0} = \frac{1}{c\sqrt{\kappa}\epsilon_0} h\left(z - \frac{c}{\sqrt{\kappa}} t\right)$. That is, the fields are scaled versions of the same pulse.