Homework 3

Due date: Tuesday, October 18

1. In our series analysis of percolation we averaged over the uniform distribution of points (graph vertices) in a volume of unspecified shape in D dimensions. We revisit this average in this problem, paying attention to some details, to help us understand why we recover the simpler random graph model in the limit $D \to \infty$.

   Our volume V is a periodic, hypercubic box of linear size L. The squared distance between a pair of points $x, y \in V$ is defined by

   $$d(x,y) = \min_{v \in (L\mathbb{Z})^D} \|x - y + v\|^2,$$

   where $\| \cdots \|^2$ is the standard squared Euclidean distance and the minimization corresponds to adding a suitable multiple of L to all the coordinate differences making each no greater than $L/2$ in magnitude (this is how one normally interprets a "periodic box"). Recall that the edge probabilities in the percolation model were functions of the squared distances.

   (a) Show that in the uniform average on three points x, y, z of two of the squared distances,

   $$\langle d(x,y)\, d(y,z) \rangle_V = \langle d(x,y) \rangle_V \, \langle d(y,z) \rangle_V,$$

   we get a result consistent with independence of the edge probabilities, as in the random graph model.

   (b) Now calculate

   $$\langle d(x,y)\, d(y,z)\, d(z,x) \rangle_V - \langle d(x,y) \rangle_V \, \langle d(y,z) \rangle_V \, \langle d(z,x) \rangle_V$$

   and show it is non-zero. However, show that one can take the limit $L, D \to \infty$ in such a way that the difference above vanishes faster than either term.

2. Recall that in the percolation model with Gaussian edge probabilities the series for the mean cluster size,

   $$\begin{aligned} S_2(d) &= \frac{1}{n} \sum_k k^2 n_k \\ &= \sum_G \frac{f_2(G)}{\sigma(G)} \frac{d^{v(G)-1}}{\tau^{D/2}(G)}, \end{aligned}$$

reduces, in the $D \to \infty$ limit, from a sum over all unlabelled connected graphs G to a sum over just tree graphs T:

$$S_2(d) \sim \sum_T \frac{f_2(T)}{\sigma(T)} d^{\,v(T)-1}, \qquad D \to \infty. \tag{1}$$

Your assignment in this problem is to show that the sum-over-cores function

$$f_2(T) = (-1)^{e(T)} \sum_{\text{cores } C \in T} (-1)^{e(C)} v^2(C)$$

vanishes unless T is a "path" — no vertices of degree greater than 2 — in which case $f_2(T) = 2$. Since $\sigma(T) = 2$ when T is a path, this shows that the term by term $D \to \infty$ limit of the series for S_2 is the same as the series $1 + d + d^2 + \cdots$ of the random graph model.

A standard method to solve problems of this type is to first construct a generating polynomial (or generating function)

$$p_T(x) = \sum_{\text{cores } C \in T} (-1)^{e(C)} x^{v(C)}.$$

The sum in question is then given by the value at $x = 1$ after differentiation and multiplication by x is applied to $p_T(x)$. The tree structure of T will allow you to express $p_T(x)$ as a product of factors, one for each leaf (degree 1 vertex) of the tree.

3. The $D \to 0$ limit of the cluster size series of the Gaussian percolation model is also simple.

   (a) Find this limit by first obtaining the $D \to 0$ limit of the cluster numbers n_k. A good starting point is the formula from lecture:

$$n_k/n = \sum_{l=0}^{\infty} \frac{\rho^{k+l-1}}{k!\, l!} \int d^D y_2 \cdots d^D y_k \int d^D z_1 \cdots d^D z_l$$
$$\sum_{C \in \mathcal{C}_k} \prod_{1 \le i_1 < i_2 \le k} P_{i_1 i_2}(C) \prod_{j=1}^{l} f(z_j).$$

   The sum is over $\mathcal{C}_k$, the set of all labeled connected graphs C on k vertices, and $P_{i_1 i_2}(C)$ is either a Gaussian or one minus a Gaussian (on the y's), respectively, if $(i_1 i_2)$ is or is not an edge in C. We defined the function f by

$$f(z) = \prod_{i=1}^{k} \left(1 - e^{-a(z-y_i)^2}\right) - 1.$$

[Hint: For $D \to 0$ the integrations that in general produce factors $\pm 1/\tau^{D/2}(C)$ simplify to ± 1. Using this fact, argue that the contribution of every graph in $\mathcal{C}_k$ is identically zero, except for the complete graph.]

(b) Using the very simple $D \to 0$ limit of the cluster numbers, find the corresponding limit of $S_2(d)$.