Homework 3

Due date: Wednesday, February 14

Lorentz invariance of the wave equation

Transform the wave equation in the "unprimed frame",

$$\frac{\partial^2\Psi}{\partial t^2} = v^2\frac{\partial^2\Psi}{\partial x^2}$$

to the "primed frame" with coordinates

$$\begin{aligned} x' &= \gamma_u\,(x - u\,t) \\ t' &= \gamma_u\left(t - (u/v^2)x\right) \ , \end{aligned}$$

where

$$\gamma_u = \frac{1}{\sqrt{1-u^2/v^2}} \ ,$$

and show that the form of the equation is unchanged. You should recognize the coordinate transformation as a boost with velocity u in a world where the maximum velocity is v.

Symmetries of the Schrödinger equation

The Schrödinger equation is a wave equation, but where the thing that's waving, written Ψ, is very mysterious:

$$i\,\hbar\,\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \ .$$

This is the Schrödinger equation that describes a particle of mass m moving in a one-dimensional world, the x-axis. The constant $\hbar$ is called "h-bar". From the presence of i, the square-root of -1, you probably guessed that Ψ is a *complex-valued function* of x and t.

1. Show that the Schrödinger equation is *not* invariant under time reversal, that is, when rewritten in terms of the variable, $t' = -t$.
2. Recall that i is a made-up number with the property that its square equals -1. Confirm that -1 times the made-up number would also have worked, as a number whose square equals -1.

3. The complex number system has a symmetry operation called *conjugation*[1]. Applying this operation just means going to the other choice of number whose square equals -1. In concrete terms, you can think of it as replacing i with $-i$ in any expression involving complex numbers. Geometrically it means that relationships among complex numbers, when represented graphically in the complex plane, are preserved when the numbers are reflected across the real axis. Demonstrate this by first rendering the complex numbers $z = 1 + 2i$ and iz as vectors in the usual complex plane where i is above the real axis. Now render the same pair of complex numbers in the reflected complex plane where i is below the real axis. What is the geometrical meaning of "multiply by i" — in both the standard complex plane and in the reflected complex plane?

4. Though the Schrödinger equation is not invariant under the naive time-reversal transformation, it *is* invariant when Ψ is transformed as well:

$$\Psi'(x, t') = \Psi^*(x, -t)$$

The conjugation symbol $*$ means replace all i's with $-i$'s. For example, $(1 + 2i)^* = 1 - 2i$. Show that you get invariance when time reversal is combined with conjugation.

5. When there is a (real-valued) potential energy $U(x)$ in the one-dimensional world, the Schrödinger equation becomes

$$i\,\hbar\,\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + U(x)\Psi \ .$$

What symmetries go away when the last term is added?

Propagating electric and magnetic fields in a co-ax

Though the exercise of representing a co-ax cable as a chain of capacitors and inductors was a nice example of model building — and also a review of the crazy voltage rule for inductors! — there is a much more direct approach that features $\mathbf{E}$ and $\mathbf{B}$ front and center. Start with the following vectorial forms of these fields, that you might have guessed from the distribution of charges and currents, in the obvious cylindrical coordinate system:

$$\begin{aligned} \mathbf{E} &= \frac{f(z,t)}{r}\,\hat{\mathbf{r}} \\ \mathbf{B} &= \frac{g(z,t)}{r}\,\hat{\phi} \ . \end{aligned}$$

[1] It's fitting that the mathematical study of symmetries, called goup theory, got its start when Évariste Galois thought to apply transformations to number systems, like the complex numbers.

At this point f and g are just a pair of arbitrary functions of two arguments and you need not worry about the applicable range of r.

1. Using small, suitably shaped Gaussian surfaces (which may be combined to form arbitrary surfaces) show that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$. Thus two of Maxwell's equations are satisfied provided you are in a region of zero charge density.

2. The computation of the curl is more complicated in the non-cartesian coordinates. Use this general formula for the curl in cylindrical coordinates,

$$\nabla \times \mathbf{V} = \left(\frac{1}{r} \frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \right) \hat{\phi} + \frac{1}{r} \left(\frac{\partial (r V_\phi)}{\partial r} - \frac{\partial V_r}{\partial \phi} \right) \hat{\mathbf{z}} \,,$$

   to work out the other two Maxwell equations for the $\mathbf{E}$ and $\mathbf{B}$ above and with the assumption that you are in a region with zero current density.

3. If you didn't make any mistakes, the two curl equations should reduce to two differential equations involving just the scalar functions f and g and the variables z and t. By taking derivatives of the two equations you can eliminate one of the functions, say g, and end up with a second-order differential equation involving just f. You have every right to suspect a conspiracy if your equation looks very familiar!

4. Make contact with the co-ax by considering the preposterous proposition that the $\mathbf{E}$ and $\mathbf{B}$ expressions apply only in the region between the two conductors, and that $\mathbf{E}$ and $\mathbf{B}$ *inside the conductors is exactly zero*! Explain why this can make sense if there is a surface charge density σ (charge/area) and surface current density $j\hat{\mathbf{z}}$ (charge/length $\times$ time) at the surfaces of the two conductors. Relate $\sigma(z,t)$, at $r = a$ and $r = b$ (as defined in lecture), to the function f, and $j(z,t)$, at $r = a$ and $r = b$ to the function g. Is charge conserved?

   Hint: Use the integral forms of the two Maxwell equations with sources to obtain σ and j. To check charge conservation, calculate the flow of charge into and out-of a small band of surface (small Δz).

5. Sketch a snapshot of $\mathbf{E}$ and $\mathbf{B}$ for the case of a pulse-like $\sigma(z,t)$ moving toward positive z with speed $c/\sqrt{\kappa}$.