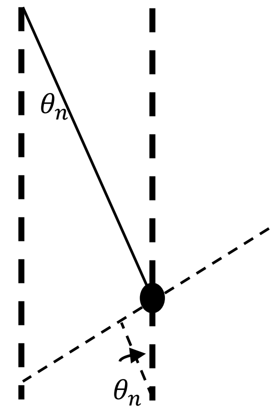
Coupled Pendulums



1. The diagram on the right might be useful. The component of the force due to gravity perpendicular to the pendulum rod is $-mg\sin(\theta_n)$, and it's negative because it acts to decrease θ_n The torque due to gravity is then

$$\tau_{ng} = -mgr\sin(\theta_n)$$

2. In class it was shown that the torque on pendulum n due to the neighboring pendulums was $\tau_n = \kappa(\theta_{n-1} - 2\theta_n + \theta_{n+1})$. The net torque on the nth pendulum is $\tau_{n,net} = -mgr\sin(\theta_n) + \kappa(\theta_{n-1} - 2\theta_n + \theta_{n+1})$. Applying Newton's second law for torques, we have that the equations of motion are

$$I\frac{d^2\theta_n}{dt^2} = -mgr\sin(\theta_n) + \kappa(\theta_{n-1} - 2\theta_n + \theta_{n+1})$$

Or equivalently,

$$\frac{d^2\theta_n}{dt^2} = -\frac{mgr}{I}\sin(\theta_n) + \frac{\kappa}{I}(\theta_{n-1} - 2\theta_n + \theta_{n+1})$$

Where I is the moment of inertia, and is a function of m and r.

3. For the case when θ_n is small, we have that $\sin(\theta_n) \approx \theta_n$. In this case, the equation becomes

$$\frac{d^2\theta_n}{dt^2} = -\frac{mgr}{I}\theta_n + \frac{\kappa}{I}(\theta_{n-1} - 2\theta_n + \theta_{n+1}).$$

4. As shown in class, the second term can be interpreted in terms of the second derivative with respect to position, and the equation becomes

$$\frac{\partial^2\theta(t,x)}{\partial t^2} = -\frac{mgr}{I}\theta(t,x) + \frac{\kappa\Delta x^2}{I}\frac{\partial^2\theta(t,x)}{\partial x^2}.$$

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1. With the assumptions we are making (i.e. constant electric field, no edge effects) the tape and metal surfaces form a parallel plate capacitor. In such a capacitor, the field is given by $E = \frac{\sigma}{\kappa\epsilon_0}$, where σ is the charge per unit area. This is simply the sum of the fields of two planes of charge. Substituting in $\sigma = \frac{Q}{Lw}$, where l and w are the length and width of the tape, we have $E = \frac{Q}{\kappa\epsilon_0 Lw}$. The potential is then $V = \int_0^d E(z)dz = \frac{Qd}{\kappa\epsilon_0 Lw}$. With this we have that the capacitance $C = \frac{Q}{V} = \frac{\kappa\epsilon_0 lw}{d}$. Finally, the capacitance per unit length is

$$\tilde{C} = \frac{\kappa\epsilon_0 w}{d}.$$

2. Ignoring the edge effects, we compute the magnetic field as the sum of the magnetic fields of two sheets of current. The current density $j = \frac{i}{w}$, the current per unit width. For an arbitrary width Δx, using Ampere's law, we have, $B_{sheet}\, 2\Delta x = \mu_0 j\Delta x = \frac{\mu_0 i\Delta x}{w}$. The field

between the sheets is then $B = \frac{\mu_0 i}{w}$. The flux is $\phi = \int_0^d \int_0^l B\, dy\, dz = \frac{\mu_0 i l d}{w}$. The inductance is $L = \frac{\phi}{i} = \frac{\mu_0 l d}{w}$. The inductance per unit length is then

$$\tilde{L} = \frac{\mu_0 d}{w}.$$

3. The velocity of waves is

$$v = \frac{1}{\sqrt{\tilde{L}\tilde{C}}} = \frac{1}{\sqrt{\frac{\kappa\epsilon_0 w}{d}\frac{\mu_0 d}{w}}} = \frac{1}{\sqrt{\kappa}}\frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{c}{\sqrt{\kappa}}.$$