Homework 2

Due date: Tuesday, October 4

1. In the random graph model we define

   $S(d) = $ size of tree a vertex belongs to, averaged over vertices in finite trees,

   where, as always, "finite" means we take the limit $n \to \infty$ while bounding k, the number of vertices in clusters/trees. Starting with the expression for $S(d)$ in terms of the cluster numbers n_k, and using duality, derive an explicit formula for $S(d)$.

2. On clear nights humans are naturally inclined to define the "constellation graph model", or CGM. The vertices of CGM are stars up to a given magnitude (the threshold of vision) and the rule for edges is that each star/vertex is connected to its nearest neighbor. A fundamental quantity in the CGM is the average number of stars S in a connected cluster — a constellation. In this problem you will determine S in the statistical ensemble where the visible stars are distributed uniformly at random in the sky, and their density is high enough that the sky may be modeled as a flat plane (rather than a sphere).

   (a) Begin by proving that all constellations in the CGM are trees, and that these trees always have a unique pair of distinguished vertices with the property that each is the other's nearest neighbor. The distinguished pair is the "core" of the constellation.

   (b) Show that
   $$S = \frac{2}{p_{\rm c}},$$
   where $p_{\rm c}$ is the probability that a randomly selected star is a core star.

   (c) Calculate $p_{\rm c}$ in the CGM and thereby determine S.

3. A very useful theorem in graph theory relates a combinatorial property of a graph, the number of spanning trees it contains, and a matrix property, the determinant of the graph's "Laplacian". In this problem we will be working with the set of connected graphs more properly called "multigraphs". In a multigraph there can be multiple edges between a pair of vertices. However, our graphs will not have "loops", which are edges joining vertices to themselves. Let $\tau(C)$ be the number of spanning trees in the multigraph C. If C has n vertices, the graph Laplacian $\Delta(C)$ is the $n \times n$ matrix whose diagonal elements are the edge degrees of the vertices and whose off-diagonal elements are the negative counts of the number of edges between pairs of vertices. Finally, let $\Delta_i(C)$ be the $(n-1) \times (n-1)$ matrix obtained by deleting row and column i of $\Delta(C)$. Kirchhoff's *matrix-tree theorem*[1] states that (for any i)

$$\tau(C) = \det \Delta_i(C).$$

   (a) Let A be an $N \times N$ real symmetric positive-definite matrix (so that it may be written as $A = B^{\mathrm{T}}B$ where B is nonsingular). Show that

   $$\int d^N x \exp\left(-x^{\mathrm{T}} A x\right) = \sqrt{\frac{\pi^N}{\det A}}.$$

   (b) We are interested in the case where $N = (n-1)D$ and A is comprised of $D \times D$ identity blocks multiplied by elements of an $(n-1) \times (n-1)$ matrix Δ_1. Show that

   $$\det A = (\det \Delta_1)^D.$$

   [Hint: Compare the eigenvalues of A and Δ_1.]

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[1]To prove the theorem (not required) you can use induction on the formula $\tau(C) = \tau(C - e) + \tau(C \circ e)$ and the corresponding statement for the graph Laplacians. Here $C - e$ is the graph C with edge e removed, while $C \circ e$ is the graph obtained by merging the two vertices connected by e.

(c) Low density expansions in statistical mechanics sometimes involve integrals of the form

$$I(C) = \int d^D x_2 \cdots \int d^D x_n \exp\left(- \sum_{(ij)\in C} (x_i - x_j)^2\right), \qquad (1)$$

where C is a connected graph of n vertices (not a multigraph in this case) and the sum runs over all edges (ij) in C. Using the previous parts of this problem, show that

$$I(C) = \left(\frac{\pi^{n-1}}{\det \Delta_1(C)}\right)^{D/2}.$$

[Hint: By translational invariance of the integrand it does not matter which coordinate is fixed and not integrated over (in equation (1) it is x_1). For the same reason, the value of the fixed coordinate does not matter so make the simplest choice: set it equal to zero.]

(d) The integral $I(C)$ is easy to evaluate when C is a tree. Perform the integral and show that $\det \Delta_1(C) = 1$ when C is a tree, a simple check of the matrix-tree theorem.

(e) The complete graph K_n is the graph on n vertices where all pairs of edges are joined by one edge. Using the matrix-tree theorem prove Cayley's formula:
$$\tau(K_n) = n^{n-2}.$$