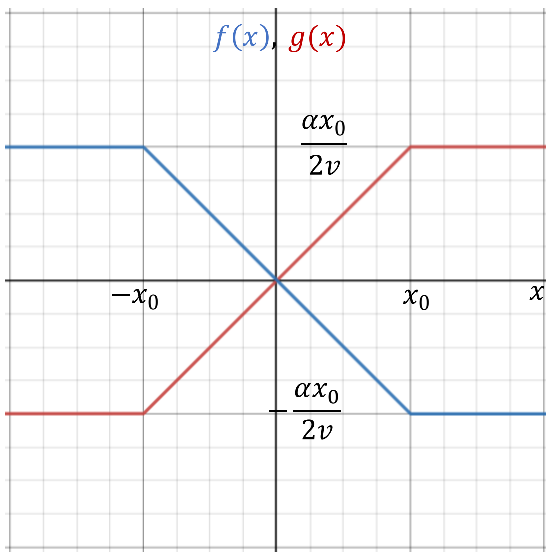
A different initial condition for torsional waves

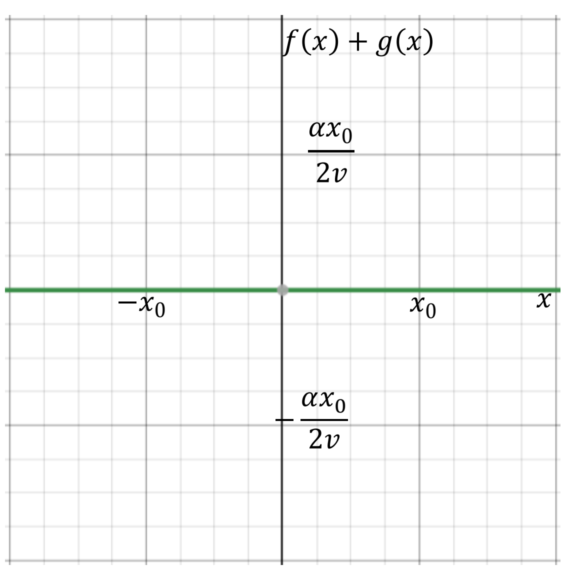
1. Recall that $f(x) = \frac{1}{2}p(x) - \frac{1}{2v}\int_0^x q(x')dx'$ and $g(x) = \frac{1}{2}p(x) + \frac{1}{2v}\int_0^x q(x')dx'$. For our case, $p(x) = \theta(x,0) = 0$, and $q(x) = \dot{\theta}(x,0) = S(x) = \begin{cases} \alpha, & |x| < x_0 \\ 0, & |x| > x_0 \end{cases}$. With this, we have

$$f(x) = -g(x) = -\frac{1}{2v}\int_0^x S(x')dx' = \begin{cases} -\dfrac{\alpha}{2v}x, & |x| < x_0 \\ -\dfrac{\alpha}{2v}x_0, & x > x_0 \\ \dfrac{\alpha}{2v}x_0, & x < -x_0 \end{cases}$$

Sketching these gives



2. Combining the above two graphs, we see the red and the blue graphs will perfectly cancel out giving a flat line.

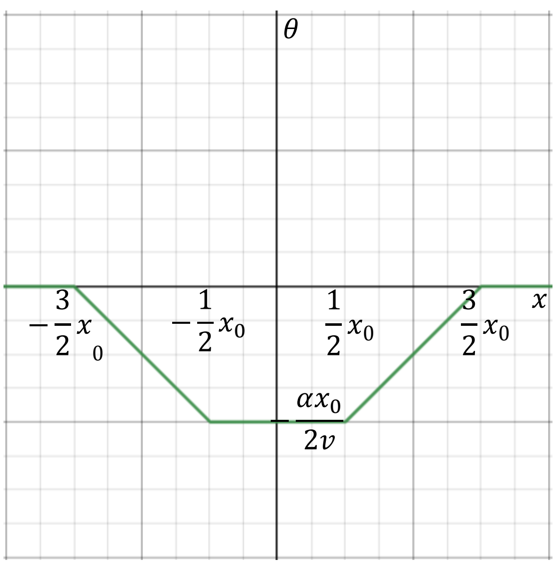


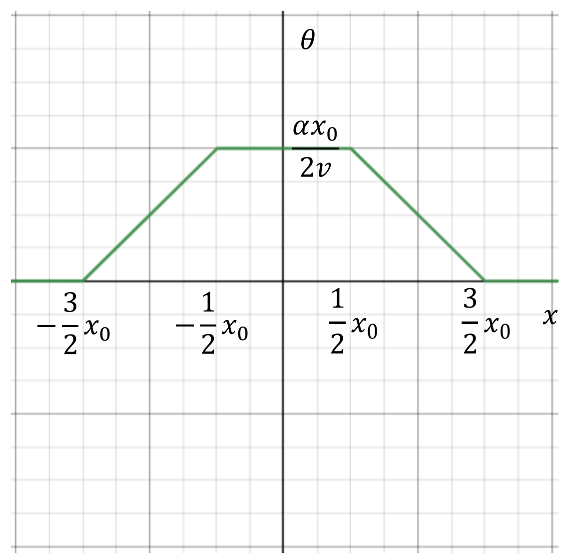
3. At $t = -\frac{1}{2}\frac{x_0}{v}$, $\theta\left(x, -\frac{1}{2}\frac{x_0}{v}\right) = f\left(x + \frac{1}{2}x_0\right) - f\left(x - \frac{1}{2}x_0\right)$. Looking at $f(x)$, we see that

$$f\left(x + \frac{1}{2}x_0\right) = \begin{cases} -\frac{\alpha}{2v}\left(x + \frac{1}{2}x_0\right), & -\frac{3}{2}x_0 < x < \frac{1}{2}x_0 \\ -\frac{\alpha}{2v}x_0, & x > \frac{1}{2}x_0 \\ \frac{\alpha}{2v}x_0, & x < -\frac{3}{2}x_0 \end{cases}$$

$$-f\left(x - \frac{1}{2}x_0\right) = \begin{cases} \frac{\alpha}{2v}\left(x - \frac{1}{2}x_0\right), & -\frac{1}{2}x_0 < x < \frac{3}{2}x_0 \\ \frac{\alpha}{2v}x_0, & x > \frac{3}{2}x_0 \\ -\frac{\alpha}{2v}x_0, & x < -\frac{1}{2}x_0 \end{cases}$$

When $x < -\frac{3}{2}x_0$, the two functions cancel out. When $-\frac{3}{2}x_0 < x < -\frac{1}{2}x_0$, the two combine into a line with slope $-\frac{\alpha}{2v}$. When $-\frac{1}{2}x_0 < x < \frac{1}{2}x_0$, the two add to $-\frac{\alpha}{2v}x_0$. When $\frac{1}{2}x_0 < x < \frac{3}{2}x_0$, we get a line with slope $\frac{\alpha}{2v}$, and finally when $x > \frac{3}{2}x_0$, the two again cancel out. This is graphed below on the left.





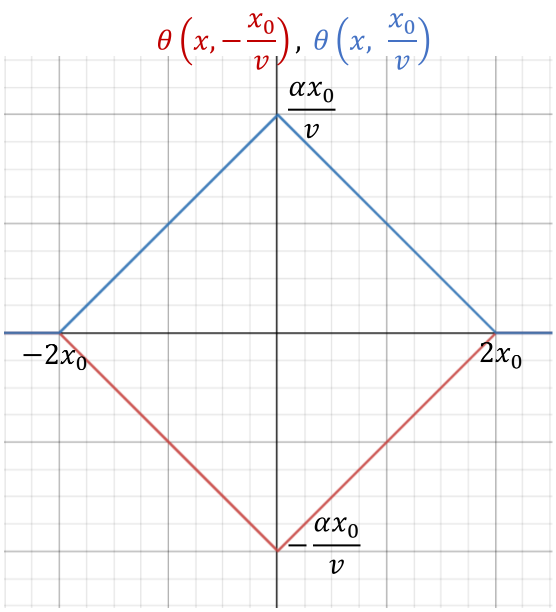
At $t = \frac{1}{2}\frac{x_0}{v}$, $\theta\left(x, \frac{1}{2}\frac{x_0}{v}\right) = f\left(x - \frac{1}{2}x_0\right) - f\left(x + \frac{1}{2}x_0\right)$. Switching the order of the terms and factoring out a -1, we get $\theta\left(x, \frac{1}{2}\frac{x_0}{v}\right) = -\left(f\left(x + \frac{1}{2}x_0\right) - f\left(x - \frac{1}{2}x_0\right)\right) = -\theta\left(x, -\frac{1}{2}\frac{x_0}{v}\right)$. The graph will then be exactly the negative as before. This is graphed above on the right.

4. At $t = \frac{x_0}{v}$, $\theta\left(x, \frac{x_0}{v}\right) = f(x - x_0) - f(x + x_0)$. At this time we have

$$f(x - x_0) = \begin{cases} -\frac{\alpha}{2v}(x - x_0), & 0 < x < 2x_0 \\ -\frac{\alpha}{2v}x_0, & x > 2x_0 \\ \frac{\alpha}{2v}x_0, & x < 0 \end{cases}$$

$$-f(x+x_0)=\begin{cases}\frac{\alpha}{2v}(x+x_0), & -2x_0 < x < 0\\ \frac{\alpha}{2v}x_0, & x>0\\ -\frac{\alpha}{2v}x_0, & x<-2x_0\end{cases}$$

For $x < -2x_0$ and $x > 2x_0$, the two will cancel out. For $-2x_0 < x < 0$, the two form a line with slope $-\frac{\alpha}{2v}$, and for $0 < x < 2x_0$, they will form a line with slope $\frac{\alpha}{2v}$. At $t = -\frac{x_0}{v}$, we get the negative of this, following the same reasoning as in part 3. The two are graphed below.

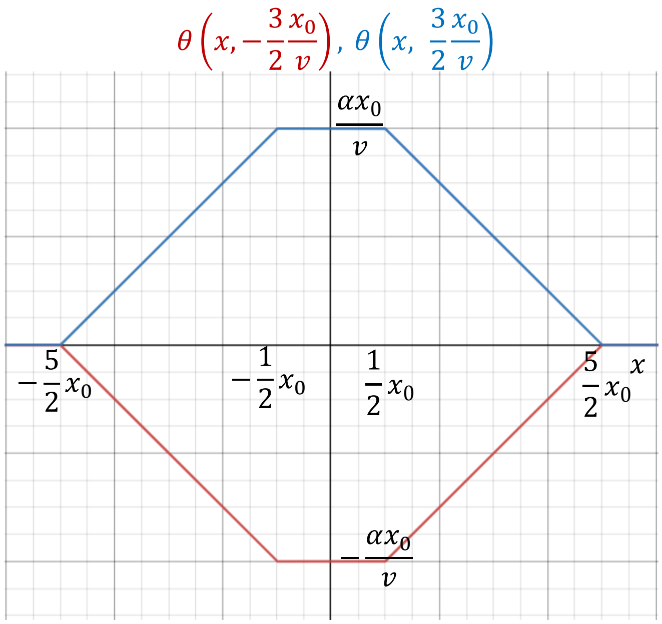


5. At $t=\frac{3}{2}\frac{x_0}{v}$, $\theta\left(x,\frac{3}{2}\frac{x_0}{v}\right)=f\left(x-\frac{3}{2}x_0\right)-f\left(x+\frac{3}{2}x_0\right)$. At this time we have

$$f(x-x_0)=\begin{cases}-\frac{\alpha}{2v}\left(x-\frac{3}{2}x_0\right), & \frac{1}{2}x_0 < x < \frac{5}{2}x_0\\ -\frac{\alpha}{2v}x_0, & x>\frac{5}{2}x_0\\ \frac{\alpha}{2v}x_0, & x<\frac{1}{2}x_0\end{cases}$$

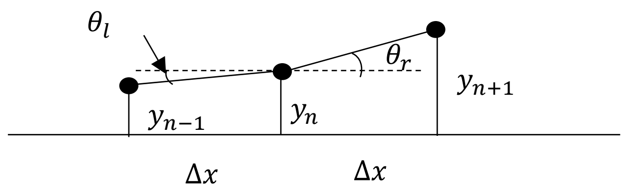
$$-f(x+x_0)=\begin{cases}\frac{\alpha}{2v}(x+x_0), & -\frac{5}{2}x_0 < x < -\frac{1}{2}x_0\\ \frac{\alpha}{2v}x_0, & x>-\frac{1}{2}x_0\\ -\frac{\alpha}{2v}x_0, & x<-\frac{5}{2}x_0\end{cases}$$

For $x < -\frac{5}{2}x_0$ and $x > \frac{5}{2}x_0$, the two will cancel out. For $-\frac{5}{2}x_0 < x < -\frac{1}{2}x_0$, the two form a line with slope $-\frac{\alpha}{2v}$, and for $\frac{1}{2}x_0 < x < \frac{5}{2}x_0$, they will form a line with slope $\frac{\alpha}{2v}$. For $-\frac{1}{2}x_0 < x < \frac{1}{2}x_0$, the two add up to $-\frac{\alpha x_0}{v}$. At $t = -\frac{3}{2}\frac{x_0}{v}$, we get the negative of all this, following the same reasoning as in part 3. The two cases are graphed below.



Transverse waves on a string

1. The little diagram on the right may be helpful in this case.
The force on the nth mass due to the $(n+1)$ mass is $T(x_{n+1})$ and directed along the line connecting the two masses. The y component of this force will then be



$$F_{n+1} = T(x_{n+1})\sin(\theta_r)$$

The force on the nth mass due to the $(n-1)$ mass is $-T(x_n)$ and it is directed along the line connecting those two masses. It's y component will be

$$F_n = -T(x_n)\sin(\theta_l),$$

where θ_l is measured counterclockwise from the horizontal between the two masses at x_n and x_{n-1} (this is so that the sign of the force will be correct in the cases when $y_n > y_{n-1}$ and $y_n < y_{n-1}$). The total vertical force is then

$$F_{n+1} + F_n = T(x_{n+1})\sin(\theta_r) - T(x_n)\sin(\theta_l)$$

Since we are assuming that the slope (and therefore the angle) is very small, we have that $\sin(\theta_r) \approx \frac{y_{n+1}-y_n}{\Delta x}$ and $\sin(\theta_l) \approx \frac{y_n-y_{n-1}}{\Delta x}$. Making this substitution, we have

$$F_{n+1} + F_n = T(x_{n+1})\frac{y_{n+1}-y_n}{\Delta x} - T(x_n)\frac{y_n-y_{n-1}}{\Delta x}.$$

2. Multiplying and dividing by Δx on the right side of the last equation in 1., we get

$$F_{n+1} + F_n = \Delta x \frac{T(x_{n+1})\frac{y_{n+1} - y_n}{\Delta x} - T(x_n)\frac{y_n - y_{n-1}}{\Delta x}}{\Delta x}.$$

Interpreting the differences in terms of derivatives, this becomes

$$F_{n+1} + F_n = \Delta x \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right).$$

3. Newton's second law applied to mass m_n is $F_{n+1} + F_n = m_n \frac{\partial^2 y_n}{\partial t^2}$. Making this substitution, and recalling that $m_n = \mu(x_n)\Delta x$, and taking the limit where x is continuous as in part 2., we find

$$\mu(x)\Delta x \frac{\partial^2 y}{\partial t^2} = \Delta x \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right).$$

Dividing both sides by Δx we finally have

$$\mu(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right).$$

4. At each point along the chain, the tension in the chain must balance the weight of the chain below that point, so $T(z) = weight\ of\ chain\ below\ z = g \times (mass\ of\ chain\ below\ z)$. The mass of the chain below z is, in general, $\int_0^z \mu(z')dz'$. For our case $\mu(z) = \mu_0$, so the integral becomes $\int_0^z \mu_0 dz = \mu_0 z$, and the tension is

$$T(z) = g\mu_0 z$$

Substituting into the final equation in part 3. (with the change $x \to z$) we get

$$\mu_0 \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}\left(g\mu_0 z\frac{\partial y}{\partial z}\right).$$

We can pull the constants $\mu_0 g$ out of the derivative and divide both sides by μ_0, giving

$$\frac{\partial^2 y}{\partial t^2} = g\frac{\partial}{\partial z}\left(z\frac{\partial y}{\partial z}\right).$$

Applying the product rule, we finally obtain

$$\frac{\partial^2 y}{\partial t^2} = g\frac{\partial y}{\partial z} + gz\frac{\partial^2 y}{\partial z^2}.$$

For the equation for torsion waves, the general solution was $f(x - vt) + g(x + vt)$, for general functions f and g. We can now check if $y = f(z - vt) + g(z + vt)$ solves the wave equation for a hanging chain. To avoid any confusion between the function g, and g the acceleration due to gravity, I will call the acceleration due to gravity a_g. Plugging in if $y = f(z - vt) + g(z + vt)$ to the left of the wave equation for the hanging chain, we get

$$\frac{\partial^2 y}{\partial t^2} = v^2\big(f''(z - vt) + g''(z + vt)\big).$$

Plugging $y = f(x - vt) + g(x + vt)$ into the right side, we get

$$a_g \frac{\partial y}{\partial z} + a_g z \frac{\partial^2 y}{\partial z^2} = a_g\big(f'(x - vt) + g'(x + vt)\big) + a_g\, z\big(f''(z - vt) + g''(z + vt)\big).$$

In general, it will not be true that $v^2\big(f''(z - vt) + g''(z + vt)\big) = a_g\big(f'(x - vt) + g'(x + vt)\big) + a_g\, z\big(f''(z - vt) + g''(z + vt)\big)$, so the general solution to the torsion wave equation will not be a solution to the hanging chain wave equation.