Homework 1

Due date: Wednesday, January 31

A different initial condition for torsional waves

In lecture we worked out the solution $\theta(x,t)$ for torsional waves when all the oscillators are initially at rest, $\dot{\theta}(x,0) = 0$, and the initial amplitudes had the shape of a triangle function, $\theta(x,0) = T(x)$. In this exercise you will work out the solution when initially the amplitudes are all zero, $\theta(x,0) = 0$, and the oscillators between $x = -x_0$ and $x = x_0$ are given a nonzero angular velocity $\alpha > 0$. We can express this initial condition as $\dot{\theta}(x,0) = S(x)$, where

$$S(x) = \begin{cases} \alpha \ , & |x| < x_0 \ , \\ 0 \ , & |x| > x_0 \ . \end{cases}$$

1. First make sketches of the functions $f(x)$ and $g(x)$. Mark all features of these functions by their values on the horizontal and vertical axes.
2. Make a sketch of $\theta(x,0) = f(x) + g(x)$ by graphically combining your graphs of f and g. Again, annotate any features in your function.
3. Make a sketch of $\theta(x,t) = f(x-vt) + g(x+vt)$, for times $t = \pm(1/2)(x_0/v)$, annotating any features in your function.
4. Repeat for $t = \pm(x_0/v)$.
5. Repeat for $t = \pm(3/2)(x_0/v)$.

Transverse waves on a string

Consider a generalization of the standard stretched string where both the mass density $\mu(x)$ and tension $T(x)$ are functions of the coordinate x along the length of the string. In this exercise you derive the equation of small-amplitude *transverse* waves on such a string. Transverse means the displacement $y(x)$ is perpendicular to the string and small-amplitude means that the "slope" of the string is small everywhere:

$$\left| \frac{dy}{dx} \right| \ll 1 \ .$$

Divide the string into small segments of length Δx and replace each segment by a mass point. The position of the n-th mass point is $x_n = n\Delta x$ and its mass is $m_n = \mu(x_n)\Delta x$. The tension acting on this mass point, due to mass point $n+1$, is

$T(x_{n+1})$. This tension creates a force toward positive x. By Newton's third law the tension acting on this same mass point, but due to mass point $n-1$, is $T(x_n)$ and the corresponding force is toward negative x.

1. When calculating the transverse motion we are only interested in the y-component of force and the resulting y-component of acceleration (the x-components of force cancel in the small amplitude approximation). Calculate the y-force F_{n+1} acting on mass point n on one side (due to $n+1$) and the y-force F_n acting on the other side (due to $n-1$). These will depend on the displacements y_{n-1}, y_n, y_{n+1} and the tensions on the two sides. Combine these to get the net force

$$F_{n+1} + F_n \tag{1}$$

   acting on mass point n. Be sure to *simplify your answer by keeping only the lowest order terms in the small amplitude approximation.*[1]

2. Interpret the differences in your answer for (1) in terms of derivatives. You should end up with

$$F_{n+1} + F_n = \Delta x \, \frac{\partial}{\partial x} \left(\cdots \right) \ ,$$

   where $\cdots$ involves $T(x)$ and $\partial y / \partial x$.

3. Write down Newton's second law for mass point n. After dividing by the common Δx factor the result will be a wave equation:

$$\mu(x) \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x} \left(\cdots \right) .$$

4. A hanging chain, or a "string" whose tension is created entirely by the weight of its links, has very interesting wave behavior! Specialize the wave equation you just derived for a chain hanging vertically from a hook on the ceiling, and with uniform mass density $\mu(z) = \mu_0$. What is $T(z)$ for this string, if $z = 0$ is the height of the lower end of the chain and $z = L$ is the height of the hook? Is the general solution of the simple wave equation (as for torsional waves) also a solution for the hanging chain?

[1] For small θ, $\sin\theta \approx \tan\theta = \text{slope}$.