Homework 1

Due date: Thursday, September 15

1. In this problem you supply some of the details that were left out of the September 1 lecture on the complexity analysis of the backtracking algorithm. Although the method of analysis is quite general, you should focus on the application to edge matching puzzles. Recall that n denotes the total number of tiles and x is the fraction of them that have been placed.

   (a) Recall that $P(x)$ is the set of all tile placements at depth x — subset selection, permutation, rotations — irrespective of the placed tiles having matched edges. Show that for large n

   $$|P(x)| \sim \left(\frac{4n}{e(1-x)^{1/x-1}}\right)^{xn}.$$

   (b) Let K denote the random variable of tile colorings. Assume that the distribution of K is such that all edge colors are independent and drawn for the same color distribution. Let $c_1(e;p,K)$ and $c_2(e;p,K)$ be the colors adjacent to edge $e \in E(x)$ for tile placement $p \in P(x)$ and coloring K. Show that

   $$\langle \prod_{e\in E(x)} \delta(c_1\,(e;p,K) = c_2(e;p,K))\,\rangle_K = q^{|E(x)|},$$

   where q is the "surprise": the probability that two edge colors (independently drawn from the same distribution) match.

   (c) From the formula (given in lecture)

   $$v(x) = \sum_{p\in P(x)} \prod_{e\in E(x)} \delta(c_1\,(e;p,K) = c_2(e;p,K))$$

   for the number of edge-matched tilings at depth x (number of nodes of the search tree at that depth), show that

   $$\begin{aligned}\langle v(x)\rangle_K &\sim \left(\frac{4nq^2}{e(1-x)^{1/x-1}}\right)^{xn} \\ &= e^{w_\alpha(x)n},\end{aligned}$$

   where the tree-width function $w_\alpha(x)$ depends on parameters only via the combination

   $$\alpha = \frac{4nq^2}{e}.$$

(d) Sketch $w_\alpha(x)$ for $0 < x < 1$ in four cases: $\alpha = 1/e$, $1/e < \alpha < 1$, $\alpha = 1$, and $1 < \alpha$. Determine the exponent $y(\alpha)$ in the backtracking complexity $e^{y(\alpha)n}$ by the number of nodes in the widest part of the tree that must be examined to find *just one branch* that leads to the solution at $x = 1$. Find the depth $x^\star$ where the tree is widest.

(e) To create a puzzle that — even in the over-constrained case $w_\alpha(1) < 0$ — is guaranteed to have a solution we use a different ensemble of colorings, with half as many independent choices such that colors on adjacent edges are matched (in a particular $p \in P(1)$). Call the random variable for such colorings K'. It is much harder to work in this ensemble because the expectation value of the product of Kroenecker delta's does not factorize and depends on the placement $p \in P(x)$. However, it is possible to evaluate

$$\langle\!\langle \delta(c_1\,(e; p, K') = c_2(e; p, K'))\,\rangle\!\rangle_{p,K'}\ ,$$

where the double angle-brackets denote an average over the matched-color ensemble and there is also a uniform average over the elements $p \in P(x)$. Show that this average equals q to leading order ($n \to \infty$ for fixed α).

2. A unitary matrix is called *Hadamard* if all its matrix elements have the same magnitude.

(a) Verify that for all $n \geq 1$, the *Fourier matrix*

$$F_{kl} = \exp\left(2\pi i k l/n\right)/\sqrt{n}$$

is an $n \times n$ Hadamard matrix.

(b) By pre- and post-multiplying a Hadamard matrix H by diagonal matrices of phases, the transformed matrix $\tilde{H}$ is still unitary and has the same element magnitudes. With this "dephasing" operation we can eliminate some continuous degrees of freedom and transform any Hadamard matrix to a standard form where all elements in its first row and column are equal to $1/\sqrt{n}$. Show that the number of remaining free variables exactly equals the naive count of the constraints satisfied by a unitary matrix. The set of dephased Hadamard matrices should thus form a discrete set, the Fourier matrices being a particular example.

(c) However, show that the continuous family of dephased matrices

$$\tilde{H} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & ie^{i\phi} & -1 & -ie^{i\phi} \\ 1 & -1 & 1 & -1 \\ 1 & -ie^{i\phi} & -1 & ie^{i\phi} \end{bmatrix}$$

for arbitrary real ϕ is Hadamard. Naive constraint counting can thus fail.