Assignment 13

Due date: **Wednesday**, December 5

*Inductance of a loop of thin wire*

Even a simple circuit formed by a battery and a light bulb has some inductance (you don’t need a solenoid). This exercise is part review, part learning a new definition, and also some practice in estimation.

Start with an infinite straight wire of radius $r$ carrying current $I$. Assume the current density is uniform over the cross section of the wire. Calculate the magnetic field everywhere (inside and outside the wire). This is a good review of Ampère’s law. Use the integrated form, with circular curves about the center of the wire.

Now take the straight wire and bend it into a closed circular loop of radius $R$. Next consider the circle formed by the center of the wire and the surface spanning it, in the shape of a disk. In this part you will estimate the magnetic flux through this surface. You can use your formula for the magnetic field of the straight wire as long as you are close enough to the wire so it still looks straight. So instead of calculating the flux through the entire disk, just calculate the flux through an annulus of width $w$ along the edge, close to the wire. When integrating you can think of the annulus as a rectangle of width $w$ and length $2\pi R$ (neglect the fact it curves). You will get an answer that depends only weakly (logarithmically) on $w$. If you set $w = R/10$ then you will still be close enough for your approximation to be good. On the other hand, you are still missing the flux through about 90 percent of the disk! This is not a problem when your wire is very thin. Suppose $r/R = 10^{-3}$ and compare the flux you get for $w = R/10$ and $w = R$ (which is really stretching the validity of your approximation and yet covers the whole disk). Based on this comparison, make an estimate of the total flux $\Phi_B$.

Define the inductance of the wire as the ratio $L = \Phi_B/I$ and use your estimate of $\Phi_B$ to find the inductance of the thin wire loop.
The moving slab of electromagnetic field

Perhaps even simpler than the sinusoidal wave described in lecture is the “moving slab”, or electromagnetic pulse. This is a propagating configuration of electric and magnetic fields, where the dynamics is purely the result of moving boundaries. The boundaries are two infinite planes: $z = a$ and $z = b$, where $a < b$ and both $a$ and $b$ are increasing with time. The electric and magnetic fields are nonzero only within the slab, i.e. the region $a < z < b$, where they are uniform: $\mathbf{E} = E \hat{x}$, $\mathbf{B} = B \hat{y}$. In this exercise you will confirm, that for a particular relationship between the constant values $E$ and $B$, and just the right boundary velocities $\dot{a}$ and $\dot{b}$, all the Maxwell equations are satisfied.

Check that the divergence equations are satisfied. This breaks down into two parts for each of the fields. Checking that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ within the slab is very easy. However, you also have to check that there are no problems at the boundaries. To do that, consider small cubic boxes that are fixed in space and straddle the boundary (fields on one half, vacuum on the other) and check that the flux into them is zero.

Check that the curl equations are satisfied. Again, this is very easy inside the slab. To check the boundaries you have to consider rectangular loops that straddle them and are fixed in space. By orienting the loops appropriately, you will be able to relate the circulation of $\mathbf{E}$ to the changing flux of $\mathbf{B}$, and vice versa. Do this at both boundaries. You will see that to satisfy both equations (at each boundary) the velocities $\dot{a}$ and $\dot{b}$ have to have a particular value and $E$ and $B$ need to be in a particular ratio.