Logarithm conventions

1. Since the energy change is $\Delta E = k_B T$, the entropy change is $\Delta S = \frac{\Delta E}{T} = k_B$. This means $S_{new} = S_{initial} + \Delta S = k_B(\log_2(\Omega) + 1) = k_B(\log_2(\Omega) + \log_2(2)) = k_B \log_2(2\Omega)$, which means $\Omega_{new} = 2\Omega$ and therefore the change in Ω is
$$\Delta\Omega = \Omega.$$
2. Following the same reasoning given in lecture, we find that $P(E_i) \propto \Omega(E - E_i) = 2^{\frac{S(E-E_i)}{k_B}}$, and $S(E - E_i) \approx S(E) - \frac{\partial S}{\partial E}|_{E_i} E_i = S(E) - \frac{E_i}{T}$, giving $2^{\frac{S(E-E_i)}{k_B}} \propto 2^{-\frac{E_i}{k_B T}}$, meaning
$$P(E_i) \propto 2^{-\frac{E_i}{k_B T}}$$
3. The ratio of one Kevin (K) to one Kelvin (K) is $\frac{1\mathrm{K}}{1K} = \frac{\left(1.38\times10^{-23}\frac{J}{\mathrm{bit}}\right)}{1.38\times10^{-23}\frac{J}{\mathrm{nat}}} = 1\frac{\mathrm{nat}}{\mathrm{bit}}$. Since $e^{\mathrm{nat}} = e$ and $e^{\mathrm{bit}} = 2 = e^{\ln(2)}$ we have that $\mathrm{nat} = 1, \mathrm{bit} = \ln(2)$, so $\frac{1\mathrm{K}}{1K} = \frac{1}{\ln(2)}$. We thus have
$$300\ K = \frac{300}{\ln(2)}\mathrm{K} = 432.8\ \mathrm{K}$$

Temperature via infinitesimal entropy changes

1. Recall that when the volume is changed slowly, $E \propto 1/V^{\frac{2}{3}}$, so $\frac{E+\Delta E_1}{E} = \frac{V^{\frac{2}{3}}}{(V-\Delta V)^{\frac{2}{3}}} = \frac{1}{\left(1-\frac{\Delta V}{V}\right)^{\frac{2}{3}}} \approx 1 + \frac{2}{3}\frac{\Delta V}{V}$, which means $\frac{\Delta E_1}{E} \approx \frac{2}{3}\frac{\Delta V}{V}$, or
$$\Delta E_1 \approx \frac{2}{3}\frac{\Delta V}{V}E.$$
2. Since the change is occurring slowly, the entropy is constant. This was one of the implications of $E \propto 1/V^{\frac{2}{3}}$, therefore
$$\Delta S_1 = 0.$$
3. Freely expanding means no forces are acting on the gas, so the work on the gas is zero, meaning
$$\Delta E_2 = 0.$$
4. For a monoatomic gas we know $\Omega \propto V^N$ for constant energy, so $S = Nk_B \ln(V) + C$, and $\Delta S_2 = S_{final} - S_{initial} = Nk_B \ln\left(\frac{V}{V-\Delta V}\right) = -Nk_B \ln\left(1 - \frac{\Delta V}{V}\right) \approx \frac{Nk_B \Delta V}{V}$.
5. The change in entropy divided by the change in energy is $\frac{\Delta S}{\Delta E} = \frac{\Delta S_2}{\Delta E_1} = \frac{\frac{Nk_B\Delta V}{V}}{\frac{2}{3}\frac{\Delta V}{V}E} = \frac{3}{2}\frac{Nk_B}{E} = \frac{1}{T}$, or
$$T = \frac{2}{3}\frac{E}{Nk_B}.$$
Rearranging the above equation we get $E = \frac{3}{2}Nk_B T$, in agreement with equipartition.

Heat Capacity of Two Simple Systems

1. For a diatomic gas, $E = \frac{5}{2}Nk_B T$, so $C = \frac{dE}{dT} = \frac{5}{2}Nk_B$, proportional to N and $\frac{C}{k_B} = \frac{5}{2}N$ is unitless.

2. For a system of magnetic dipoles in a magnetic field, $E = -\frac{Nb^2}{k_B T}$, so $C = \frac{dE}{dT} = \frac{Nb^2}{k_B T^2}$, which is proportional to N, and $\frac{C}{k_B} = \frac{Nb^2}{k_B^2 T^2}$ is unitless, since b and $k_B T$ have units of energy.

Precision of the entropy maximum

1. We know that for a monoatomic gas at constant volume $\Omega \propto E^{\frac{3}{2}N}$, so for the combined system we have
$$\Omega = \Omega_1(E + \Delta E) \times \Omega_2(E - \Delta E) \propto (E + \Delta E)^{\frac{3}{2}N}(E - \Delta E)^{\frac{3}{2}N}.$$
Factoring out an $E^{\frac{3}{2}N}$ from each factor we have $\Omega(\Delta E) \propto E^{3N}\left(1 + \frac{\Delta E}{E}\right)^{\frac{3}{2}N}\left(1 - \frac{\Delta E}{E}\right)^{\frac{3}{2}N} = E^{3N}\left(1 - \frac{\Delta E^2}{E^2}\right)^{\frac{3N}{2}}$, but E^{3N} is simply a constant, so
$$\Omega(\Delta E) \propto \left(1 - \frac{\Delta E^2}{E^2}\right)^{\frac{3N}{2}}.$$
2. We can rewrite our last expression as $\Omega(\Delta E) \propto \left(1 - \frac{\frac{3}{2}N\left(\frac{\Delta E}{E}\right)^2}{\frac{3}{2}N}\right)^{\frac{3N}{2}}$. Letting $M := \frac{3}{2}N$, we have $\Omega(\Delta E) \propto \left(1 - \frac{\frac{3}{2}N\left(\frac{\Delta E}{E}\right)^2}{M}\right)^{M}$. Now for large N which means large M we get $\Omega(\Delta E) \propto e^{-\frac{3}{2}N\left(\frac{\Delta E}{E}\right)^2}$, as long as $\frac{3}{2}N\left(\frac{\Delta E}{E}\right)^2$ does not also diverge, which it won't, because E will grow as N grows.
3. Letting $\epsilon = \frac{\Delta E}{E}$, we can write our previous result as
$$\Omega(\Delta E) \propto e^{-\frac{3}{2}N\epsilon^2} = e^{-\frac{1}{2}\frac{\epsilon^2}{(1/3N)}} = e^{-\frac{1}{2}\left(\frac{\epsilon}{1/3N}\right)^2}.$$
Which means $\sigma = \frac{1}{\sqrt{3N}} \approx 7.4 \times 10^{-13}$.