

Homework 11

Due date: Wednesday, May 1

Particle position distribution in a 3D potential

Consider a point particle confined to a 3D box and subject to a position-dependent potential energy $U(\mathbf{r})$. Calculate the probability density $\rho(\mathbf{r})$ of the particle's position by assuming that the particle's total energy E is everywhere greater than the potential energy, and that the ergodic hypothesis applies. The probability density is normalized so that

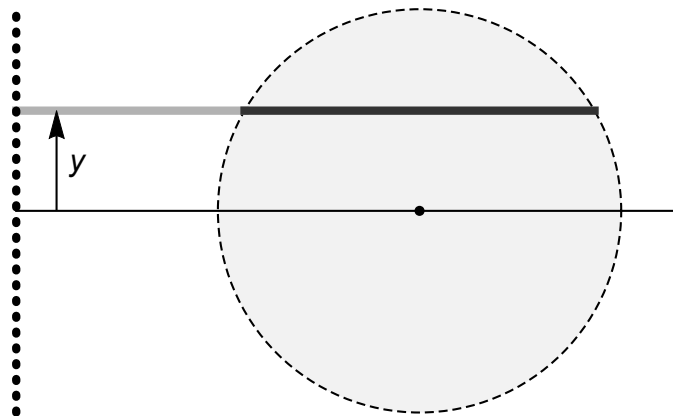
$$\int_{\text{box}} \rho(\mathbf{r}) d^3\mathbf{r} = 1 .$$

Tracer particle analysis of soft billiards

The ergodic hypothesis is difficult to prove, even for simple systems. This exercise should at least make the hypothesis plausible for the soft billiards system.

Instead of following a single very complex trajectory, we will analyze simple families of trajectories over a limited time. The family we have in mind is best described as a set of “tracer” particles, initially arranged with uniform density ρ along the y -axis. All particles are in the low potential (“fast”) region and have velocity $v_1 \hat{x}$.

First consider the total time T_1 spent by the tracer particles, in region 1, when crossing a circular region of radius r :



Set this up by first showing that the tracer particles with “impact parameter” y spend time

$$t_1(y) = \frac{2\sqrt{r^2 - y^2}}{v_1}$$

in the circular region. Then compute the integral

$$T_1 = \int_{-r}^{+r} t_1(y)(\rho dy)$$

to find the time spent by all the tracer particles in the circular region.

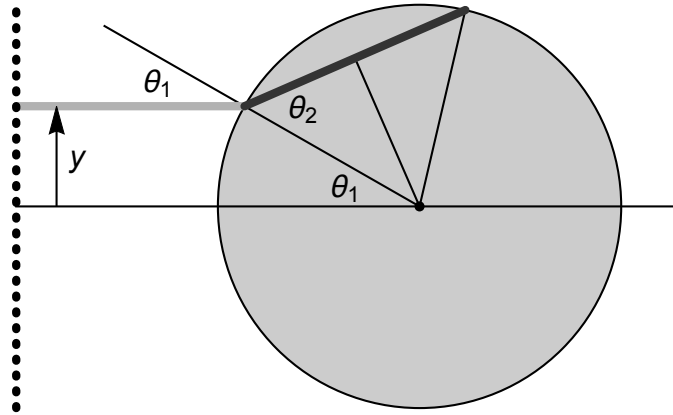
Hint: The integral is just the area of a familiar shape.

Next suppose the parallel streaming tracer particles encounter a circular region of higher potential, where their speed slows to v_2 . This region also has radius r , so we can compare with the earlier calculation. The time spent by all the tracer particles crossing this region can again be written as an integral,

$$T_2 = \int_{-r}^{+r} t_2(y)(\rho dy),$$

where $t_2(y)$ is the time spent by a tracer particle with impact parameter y , as before.

Now use Schnell’s law for particles and this diagram



to show that

$$t_2(y) = \begin{cases} \left(\frac{2r}{v_2}\right) \sqrt{1 - \left(\frac{v_1}{v_2}\right)^2 \left(\frac{y}{r}\right)^2}, & |y| < \left(\frac{v_2}{v_1}\right) r \\ 0, & \text{otherwise.} \end{cases}$$

Explain why the integrand is zero at the larger impact parameters.

Finally, evaluate the integral for T_2 and observe that it exactly equals T_1 !

Hint: After a change of variable this integral is identical to the one you calculated earlier.

Monoatomic and diatomic gases in thermal contact

Consider a monoatomic gas comprising N_1 atoms of mass m_1 , confined to a box of volume V_1 , and having total energy E_1 . Also consider a diatomic gas comprising N_2 molecules¹ of mass m_2 , moment of inertia I , confined to a box of volume V_2 , and having total energy E_2 . In the monoatomic gas the energy of a single particle (atom) is

$$\frac{1}{2m_1}(p_x^2 + p_y^2 + p_z^2).$$

A single particle (molecule) in the diatomic gas has energy

$$\frac{1}{2m_2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2I}(L_a^2 + L_b^2),$$

where L_a and L_b are the angular momenta about two orthogonal axes perpendicular to the instantaneous axis of the molecule.

Try to answer all of the following questions **without using formulas**. Microstate-counting is a valuable skill and is actually free of complicated details when you are allowed — as in this problem — to ignore constant factors. You may also assume N_1 and N_2 are very very large.

1. Up to a constant factor, calculate the number of microstates of the monoatomic gas, $\Omega_1(V_1, E_1)$.
2. Up to an additive constant, calculate the entropy of the monoatomic gas, $S_1(V_1, E_1)$.
3. Calculate the temperature of the monoatomic gas, T_1 .
4. Up to a constant factor, calculate the number of microstates of the diatomic gas, $\Omega_2(V_2, E_2)$.
5. Up to an additive constant, calculate the entropy of the diatomic gas, $S_2(V_2, E_2)$.
6. Calculate the temperature of the diatomic gas, T_2 .
7. Now suppose the two boxes of gas are placed next to each other, and the partition between them allows the transfer of energy as heat. In other words, the two systems are in thermal contact. Calculate the energy transferred ΔE , so that the gases, now with energies $E_1 + \Delta E$ and $E_2 - \Delta E$, are in thermal equilibrium. Express ΔE only in terms of N_1 , N_2 , T_1 and T_2 .
8. Does the sign of ΔE agree with the notion that “heat flows from hot to cold”?

¹Do not confuse this with nitrogen molecules. The symbol N_2 denotes the number of molecules in the second box.