Schnell's law for particles

1. Schnell's law states $|k_1|\sin(\theta_1) = |k_2|\sin(\theta_2)$. The phase velocity is given by $v_\phi = \frac{\omega}{k}$, which means $k = \frac{\omega}{v_\phi}$. By conservation of energy, we know ω is a constant, so Schnell's law is
$$\frac{\sin(\theta_1)}{v_{\phi,1}} = \frac{\sin(\theta_2)}{v_{\phi,2}}.$$
2. The group velocity is $v_g = \frac{\partial\omega}{\partial k} = 2Ak$, which means $k = \frac{v_g}{2A}$, and $A = \frac{\hbar}{2m} = constant$, so Schnell's law is
$$v_{g,1}\sin(\theta_1) = v_{g,2}\sin(\theta_2)$$

Reflection and transmission of a particle wave

1. The matching conditions are the same as those given in lecture, so the transmission and reflection coefficients are also given by the same equations:
$$\tilde{R} = \frac{k_1 - k_2}{k_1 + k_2}, \tilde{T} = \frac{2k_1}{k_1 + k_2}.$$
For this case, we have $k_1 = \sqrt{\frac{\hbar\omega - U_1}{\hbar A}}$, and $k_2 = \sqrt{\frac{\hbar\omega - U_2}{\hbar A}}$. Substituting into the expressions for $\tilde{R}$ and $\tilde{T}$, and multiplying the numerator and denominator by $\sqrt{\hbar A}$, we have
$$\tilde{R} = \frac{\sqrt{\hbar\omega - U_1} - \sqrt{\hbar\omega - U_2}}{\sqrt{\hbar\omega - U_1} + \sqrt{\hbar\omega - U_2}}, \tilde{T} = \frac{2\sqrt{\hbar\omega - U_1}}{\sqrt{\hbar\omega - U_1} + \sqrt{\hbar\omega - U_2}}.$$
For case a, the transmission and reflection probabilities are simply the squares of the above expressions.
2. In case b, we have that $\hbar\omega < U_2$, which means $k_2 = \sqrt{\frac{\hbar\omega - U_2}{\hbar A}} = i\sqrt{\frac{U_2 - \hbar\omega}{\hbar A}}$ is purely imaginary. In this case, the coefficients become
$$\tilde{R} = \frac{\sqrt{\hbar\omega - U_1} - i\sqrt{U_2 - \hbar\omega}}{\sqrt{\hbar\omega - U_1} + i\sqrt{U_2 - \hbar\omega}}, \tilde{T} = \frac{2\sqrt{\hbar\omega - U_1}}{\sqrt{\hbar\omega - U_1} + i\sqrt{U_2 - \hbar\omega}}.$$
And the wavefunction in the region with potential energy U_2 is $\tilde{\psi} = \tilde{T}e^{ik_2 y} = \tilde{T}e^{i\left(i\sqrt{\frac{U_2 - \hbar\omega}{\hbar A}}\right)} = \tilde{T}e^{-\sqrt{\frac{U_2 - \hbar\omega}{\hbar A}}}$, which is evanescent. The reflection probability is
$$|R|^2 = \tilde{R}\tilde{R}^* = \left(\frac{\sqrt{\hbar\omega - U_1} - i\sqrt{U_2 - \hbar\omega}}{\sqrt{\hbar\omega - U_1} + i\sqrt{U_2 - \hbar\omega}}\right)\left(\frac{\sqrt{\hbar\omega - U_1} + i\sqrt{U_2 - \hbar\omega}}{\sqrt{\hbar\omega - U_1} - i\sqrt{U_2 - \hbar\omega}}\right) = 1.$$

Sound at an interface

1. We know $s_<(y,t) = e^{ik_1 y} + Re^{-ik_1 y}, S_>(y,t) = Te^{ik_2 y}$. Matching them at $y = 0$ gives
$$1 + R = T.$$
2. For this case, $\nabla \cdot \boldsymbol{s} = \frac{\partial s}{\partial y}$, from which we have $\frac{\partial s}{\partial y} = -\frac{\delta p}{B}$, or $\delta p = -B\frac{\partial s}{\partial y}$. Since the pressure perturbation on either side of the interface must be equal, this means $B_1\frac{\partial s_<}{\partial y}|_{y=0} = B_2\frac{\partial s_>}{\partial y}|_{y=0}$, which implies

$$k_1 B_1 (1 - R) = k_2 B_2 T.$$

3. Substituting the result from part 1 into that from part 2 we have $k_1 B_1 (1 - R) = k_2 B_2 (1 + R)$, which gives

$$R = \frac{k_1 B_1 - k_2 B_2}{k_1 B_1 + k_2 B_2} = \frac{B_1 - \frac{k_2}{k_1} B_2}{B_1 + \frac{k_2}{k_1} B_2},$$

From $v_1 k_1 = v_2 k_2$ we have $\frac{k_2}{k_1} = \frac{v_1}{v_2}$, so

$$R = \frac{B_1 - \frac{v_1}{v_2} B_2}{B_1 + \frac{v_1}{v_2} B_2} = \frac{\frac{B_1}{v_1} - \frac{B_2}{v_2}}{\frac{B_1}{v_1} + \frac{B_2}{v_2}} = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2},$$

And substituting into the result of part 2 and simplifying,

$$T = \frac{2\beta_1}{\beta_1 + \beta_2}.$$

4. We know $I = \ Bv \left(\frac{\partial s}{\partial y}\right)^2$ which at $y = 0$ is $-Bk^2 v C^2 = -\beta \omega^2 C^2$ where C is $1, R, T$ for the incident, reflected and transmitted wave respectively. The sum of the reflected and transmitted intensities is then

$$I_r + I_t = -\beta_1 \omega^2 R^2 - \beta_2 \omega^2 T^2$$

$$= -\omega^2 \left(\beta_1 \left(\frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \right)^2 + \beta_2 \left(\frac{2\beta_1}{\beta_1 + \beta_2} \right)^2 \right)$$

$$= -\frac{\omega^2 \beta_1}{(\beta_1 + \beta_2)^2} (\beta_1^2 + 2\beta_1 \beta_2 + \beta_2^2)$$

$$= \beta_1 \omega^2 \frac{(\beta_1 + \beta_2)^2}{(\beta_1 + \beta_2)^2} = I_i.$$