Homework 10

Due date: Wednesday, April 24

Schnell's law for particles

When modeled as a wave, a particle of mass m moving in a 2D region of potential energy U has dispersion relation

$$\omega(\mathbf{k}) = A\,k^2 + B\ ,$$

where $k^2 = k_x^2 + k_y^2$ and the constants

$$A = \frac{\hbar}{2m} \qquad B = \frac{U}{\hbar}$$

are related to m and U by $\hbar$, or Planck's constant divided by 2π.

Consider a particle/wave that encounters a change in potential energy. The incident wave is in a region with $U = U_1$ and after crossing a planar interface is in a region of potential energy $U = U_2$. The wave's wave-vectors in the two regions, $\mathbf{k}_1$ and $\mathbf{k}_2$, make angles θ_1 and θ_2 with the normal to the interface.

1. Express Schnell's law, the relation between θ_1 and θ_2, in terms of the wave's **phase velocities** in the two regions, $v_{\phi 1}$ and $v_{\phi 2}$.
2. Express Schnell's law, the relation between θ_1 and θ_2, in terms of the wave's **group velocities** in the two regions, v_{g1} and v_{g2} .

Reflection and transmission of a particle wave

Using the same particle/wave model as in the previous problem, calculate the transmission and reflection amplitudes, T and R, of a normally incident ($\theta_1 = 0$) wave in two cases:

(a) $\hbar\omega = \hbar A\,k_1^2 + U_1 > U_2$

(b) $\hbar\omega = \hbar A\,k_1^2 + U_1 < U_2$

Here are some things to keep in mind:

- In case (a) the total energy of the incident particle/wave in region 1 is greater than the potential energy in region 2. Classically we know transmission is allowed by energy conservation.

- In case (b) the total energy of the incident particle/wave in region 1 is less than the potential energy in region 2. Classically we know transmission is impossible.
- Set the amplitude of the incident wave equal to 1.
- T and R are in general complex numbers (the combination of a real amplitude with a phasor).
- Though T and R are complex, their squared-magnitudes $|T|^2$ and $|R|^2$ are related to the probability the incident particle is transmitted or reflected.
- As in lecture, the boundary condition at the interface is that both the wave and its normal derivative are continuous.
- You will find that a solution for case (b) is possible only if the wave in region 2 is evanescent.
- Though $|R| > 0$ for case (a) seems strange classically, this is explained by the fact that U changes abruptly at the interface — on a scale small compared to the particle's wavelength.
- Be sure to check that $|R| = 1$ in case (b) !

Sound at an interface

Unlike water-surface waves (lecture) and particle waves (previous problem), the boundary condition for sound at the interface of two media (e.g. air and water) is slightly more complicated. For particle waves the boundary condition was to equate both the wave amplitude and its normal derivative at the interface. This exercise will show you how this is modified in the case of sound. You will also see how energy (intensity) is conserved in the transmission/reflection process.

Consider normal incidence, in the positive y-direction. As usual, the incident wave in the region $y < 0$ has unit amplitude,

$$s_< = e^{ik_1 y} + Re^{-ik_1 y} \ ,$$

and the time-dependence phasor $e^{-i\omega t}$, common to all the waves, is left out. In the region $y > 0$ there is only a transmitted wave:

$$s_> = Te^{ik_2 y} \ .$$

The two wave vectors are related to the common frequency and the phase velocities in the two media in the usual way:

$$\omega = v_1 k_1 = v_2 k_2 \ .$$

1. The wave amplitude s is the vector displacement of groups of particles, always in the y direction for our normal incidence geometry,

$$\mathbf{s}(y,t) = s(y,t)\hat{\mathbf{y}} \ ,$$

where s is either the function $s_<$ or $s_>$, depending on the sign of y. At a water-air interface, for example, (the real part of) $s_<(0,t)$ is the displacement of the surface of the water, while (the real part of) $s_>(0,t)$ is the displacement of the surface of the air. But these two surfaces coincide, so

$$s_<(0,t) = s_>(0,t)$$

Use this to determine one equation for R and T.

2. The pressures on the two sides of the interface must also be equal at all times, otherwise particles near the interface would experience infinite acceleration! Use the relation you previous derived,

$$\nabla \cdot \mathbf{s} = -\frac{\delta p}{B} \ ,$$

to obtain a second equation for R and T. This equation will involve both the wave vectors and the bulk moduli (B_1 and B_2) in the two media.

3. Solve your two equations for R and T. Express these by only using the combinations $\beta_1 = B_1/v_1$ and $\beta_2 = B_2/v_2$.

4. Check conservation of energy:

$$I_i = I_r + I_t \ .$$

The three terms are the intensities of the three waves (incident, reflected, and transmitted). From lecture,

$$I_i = B_1 v_1 \left(\frac{\partial s_i}{\partial y}\right)^2 \ ,$$

where

$$s_i = \mathrm{Re}\left[e^{ik_1 y - i\omega t}\right] \ ,$$

and similarly for the other two intensities. The algebra (for checking energy conservation) is simplest if you work with the ratios β_1 and β_2.