Assignment 10

Due date: Thursday, November 16

Angular momentum content of the magnetic field of a trap

A static magnetic field $\mathbf{B}$, such as used in an atom-trap, can always be expressed as $\nabla \Phi$, where the magnetic potential function $\Phi$ satisfies the Laplace equation. A systematic expansion of $\Phi$ uses the fact that the functions

$$
\begin{align*}
x \pm iy &= r \sin \theta \, e^{\pm i \phi} \\
z &= r \cos \theta
\end{align*}
$$

are a basis for the angular momentum functions for $l = 1$. By the addition rule of angular momentum, all angular momentum functions (spherical harmonics) with angular momentum $l$ and below can therefore be expressed as polynomials in $x$, $y$, and $z$ of degree at most $l$.

Show that a $\Phi$ of polynomial degree 2 ($l \leq 2$) can never produce a magnetic field having the necessary properties of a magnetic trap:

- $|\mathbf{B}|$ has a local minimum$^\dagger$.
- At the minimum, $|\mathbf{B}| > 0$.

Basis change to align the field

The interaction of a spin-1/2 particle with a magnetic field $\mathbf{B}$ is given by the Hamiltonian term

$$
\mu \mathbf{B} \cdot \mathbf{\sigma},
$$

where $\mu$ is the particle’s magnetic dipole moment and $\mathbf{\sigma}$ is the vector of Pauli matrices that make up the spin operator. To better understand the trapping power of a designed, spatially varying magnetic field, we make a basis change (dependent on position) so that “spin-up” always corresponds to the direction of the magnetic field. The required unitary transformation is the following

$$
U = \frac{1}{\sqrt{2}} \left( \sqrt{1 + \hat{\mathbf{b}} \cdot \mathbf{\hat{z}}} \, \mathbb{1} - i \frac{\hat{\mathbf{b}} \times \mathbf{\hat{z}} \cdot \mathbf{\sigma}}{\sqrt{1 + \hat{\mathbf{b}} \cdot \mathbf{\hat{z}}}} \right),
$$

$^\dagger$Here “local minimum” means the minimum occurs at a point as opposed to an entire line or plane.
where $\hat{b} = B/|B|$. By repeatedly using the Pauli-matrix identity (for general vectors $a$ and $b$)

$$(a \cdot \sigma)(b \cdot \sigma) = a \cdot b \mathbb{1} + i a \times b \cdot \sigma,$$

show that

$$U B \cdot \sigma U^\dagger = |B|\sigma_z.$$

**Artificial vector potential for the magnetic trap**

After making the basis change in the previous problem, the Hamiltonian for the spin-$1/2$ particle is given by

$$H = -\frac{\hbar^2}{2m}(\nabla + A) \cdot (\nabla + A) + \mu |B|\sigma_z$$

$$= -\frac{\hbar^2}{2m}\nabla^2 + \mu |B|\sigma_z + H_{\text{int}},$$

where $m$ is the particle mass and

$$A = U(\nabla U^\dagger)$$

is an artificial vector potential, each vector-component of which is a spin operator (a combination of Pauli matrices). If we could neglect $H_{\text{int}}$, then a spin-up particle would be permanently trapped in a trap where $|B|$ satisfies the properties given in the first problem. The interaction

$$H_{\text{int}} = -\frac{\hbar^2}{2m}((\nabla \cdot A) + A \cdot A + 2A \cdot \nabla)$$

has two terms that could bring about a transition from spin-up to spin-down. Calculate $F$ and $G$, keeping only the lowest non-vanishing terms in an expansion in powers of $x$, $y$ and $z$ (i.e. the combined power of $r$, the distance to the center of the trap). Use the three-parameter expression for $B$ derived in lecture:

$$B = \left(\beta x - \frac{\gamma}{2} xz\right) \hat{x} + \left(-\beta y - \frac{\gamma}{2} yz\right) \hat{y} + \left(B - \frac{\gamma}{4}(x^2 + y^2) + \frac{\gamma}{2} z^2\right) \hat{z},$$

$B$, $\beta$ and $\gamma$ are positive and satisfy

$$\frac{\beta^2}{B} > \frac{\gamma}{2}.$$

You should pay special attention to the off-diagonal terms, as they are responsible for the transition.

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2You may use Mathematica for this problem.