Assignment 10

Due date: Thursday, November 16

Angular momentum content of the magnetic field of a trap

A static magnetic field $\mathbf{B}$, such as used in an atom-trap, can always be expressed as $\nabla\Phi$, where the magnetic potential function Φ satisfies the Laplace equation. A systematic expansion of Φ uses the fact that the functions

$$x \pm iy = r \sin\theta \, e^{\pm i\phi}$$
$$z = r\cos\theta$$

are a basis for the angular momentum functions for $l = 1$. By the addition rule of angular momentum, all angular momentum functions (spherical harmonics) with angular momentum l and below can therefore be expressed as polynomials in x, y, and z of degree at most l.

Show that a Φ of polynomial degree 2 ($l \leq 2$) can never produce a magnetic field having the necessary properties of a magnetic trap:

- $|\mathbf{B}|$ has a local minimum[1].
- At the minimum, $|\mathbf{B}| > 0$.

Basis change to align the field

The interaction of a spin-1/2 particle with a magnetic field $\mathbf{B}$ is given by the Hamiltonian term

$$\mu \, \mathbf{B} \cdot \boldsymbol{\sigma},$$

where μ is the particle's magnetic dipole moment and $\boldsymbol{\sigma}$ is the vector of Pauli matrices that make up the spin operator. To better understand the trapping power of a designed, spatially varying magnetic field, we make a basis change (dependent on position) so that "spin-up" always corresponds to the direction of the magnetic field. The required unitary transformation is the following

$$U = \frac{1}{\sqrt{2}} \left(\sqrt{1 + \hat{\mathbf{b}} \cdot \hat{\mathbf{z}}} \; \mathbb{1} - i \frac{\hat{\mathbf{b}} \times \hat{\mathbf{z}} \cdot \boldsymbol{\sigma}}{\sqrt{1 + \hat{\mathbf{b}} \cdot \hat{\mathbf{z}}}} \right),$$

[1] Here "local minimum" means the minimum occurs at a point as opposed to an entire line or plane.

where $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$. By repeatedly using the Pauli-matrix identity (for general vectors $\mathbf{a}$ and $\mathbf{b}$)

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b}\ \mathbb{1} + i\,\mathbf{a}\times\mathbf{b}\cdot\boldsymbol{\sigma},$$

show that

$$U\ \mathbf{B}\cdot\boldsymbol{\sigma}\ U^\dagger = |\mathbf{B}|\sigma_z.$$

Artificial vector potential for the magnetic trap[2]

After making the basis change in the previous problem, the Hamiltonian for the spin-1/2 particle is given by

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}(\nabla + \mathbf{A})\cdot(\nabla + \mathbf{A}) + \mu\,|\mathbf{B}|\sigma_z \\ &= -\frac{\hbar^2}{2m}\nabla^2 + \mu\,|\mathbf{B}|\sigma_z + H_{\text{int}}, \end{aligned}$$

where m is the particle mass and

$$\mathbf{A} = U(\nabla\, U^\dagger)$$

is an artificial vector potential, each vector-component of which is a spin operator (a combination of Pauli matrices). If we could neglect H_{int}, then a spin-up particle would be permanently trapped in a trap where $|\mathbf{B}|$ satisfies the properties given in the first problem. The interaction

$$\begin{aligned} H_{\text{int}} &= -\frac{\hbar^2}{2m}((\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot\mathbf{A} + 2\mathbf{A}\cdot\nabla) \\ &= F + \mathbf{G}\cdot\nabla \end{aligned}$$

has two terms that could bring about a transition from spin-up to spin-down. Calculate F and $\mathbf{G}$, **keeping only the lowest non-vanishing terms** in an expansion in powers of x, y and z (i.e. the combined power of r, the distance to the center of the trap). Use the three-parameter expression for $\mathbf{B}$ derived in lecture:

$$\mathbf{B} = \left(\beta x - \frac{\gamma}{2}xz\right)\hat{\mathbf{x}} + \left(-\beta y - \frac{\gamma}{2}yz\right)\hat{\mathbf{y}} + \left(B - \frac{\gamma}{4}(x^2+y^2) + \frac{\gamma}{2}z^2\right)\hat{\mathbf{z}},$$

B, β and γ are positive and satisfy

$$\frac{\beta^2}{B} > \frac{\gamma}{2}.$$

You should pay special attention to the off-diagonal terms, as they are responsible for the transition.

[2] You may use *Mathematica* for this problem.