Assignment 9

Due date: Friday, March 31

Hamiltonian dynamics of the hydrogen atom

As an alternative to the usual circular-orbit Bohr model of the hydrogen atom, suppose the electron moves on a *single axis* that passes through the proton. The Hamiltonian for this system is

$$\mathcal{H}(x,p) = \frac{p^2}{2m} - \frac{A}{|x|},$$

where x is the electron position along the axis, p is the conjugate momentum, m is the electron mass, and $A = e^2/(4\pi\epsilon_0)$ in MKS.

Write down Hamilton's equations of motion for $\mathcal{H}(x,p)$.

Consider a periodic orbit where, at $t = 0$, the electron is at rest at $x = x_0 > 0$. Its energy on this orbit will be $E = -A/x_0$. After one-quarter of the orbital period, $t = T/4$, the electron will have just passed through the proton at $x = 0$. Later, after reaching $x = -x_0$ at $t = T/2$, the electron reverses its direction and retraces its motion back to $x = x_0$ to complete one period. Sketch one complete orbit in phase space.

For the orbit with energy $\mathcal{H} = E = -A/x_0$, calculate the action (phase-space area) enclosed by the orbit, $I = \oint p\, dx/2\pi$. Helpful fact:

$$\int_0^{x_0} \sqrt{(1/x) - (1/x_0)}\, dx = \pi\sqrt{x_0}/2.$$

Express the energy $E = -A/x_0$ of the orbit only in terms of I (and constants) as $E = -B/I^2$; find the value of the constant B. Notice that this energy expression is consistent in form with the Rydberg series if I is quantized in integer multiples of Planck's constant.

Canonical transformation for position-dependent mass

The Hamiltonian below describes a free particle with a position-dependent mass ($m \propto 1/q$):

$$\mathcal{H}(q,p) = Cqp^2.$$

In this problem you will simplify this Hamiltonian by applying the canonical transformation generated by $F(q,Q) = q/Q$.

First obtain the canonical transformation $Q = Q(q,p)$, $P = P(q,p)$ and the transformed Hamiltonian, $\mathcal{H}'(Q,P)$.

Next, write down Hamilton's equations for Q and P and their most general solution.

Finally, using your general solution for Q and P, write down the most general solution for the original position variable, q.

Ergodicity in the simple Yang-Mills model

Recall the Hamiltonian describing a special mode of motion of Yang-Mills fields (lecture 21):

$$\mathcal{H} = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + x^2y^2 + \tfrac{1}{2}A(x^2+y^2).$$

The last term, added by hand, prevents x or y from running off to infinity in numerical work. By keeping A small, the behavior of the system in a bounded range of the origin is only weakly altered.

As a result of the randomness in the position and momentum of the "particle" whenever it is ejected from one of the four potential-energy "canyons", this system has strongly mixing dynamics and should satisfy the ergodic hypothesis. A related but different question concerns the distribution of just the particle's position. Do you expect it to spend more time where the potential is low, or where the potential is high? Formulate a hypothesis on this question and support it by intuition, math, general principles, etc.

Test your hypothesis by numerically computing a long trajectory and stroboscopically plotting the points $x(t), y(t)$ at times t spaced at intervals $\Delta t = 1$. Set $A = 0.1$ and choose as your initial conditions

$$x(0) = 1 \quad y(0) = 0 \quad p_x(0) = 0 \quad p_y(0) = 2.$$

You should have a good picture of the distribution by the time you have plotted 5000 points. Use a differential equation solver (odeint, NDSolve) to avoid the accumulation of finite time-step errors.