Assignment 8

Due date: Monday, March 27

Hamilton's principle with Hamiltonian variables

Hamilton's extremal action principle is usually presented with Lagrangian variables, but it has a nice counterpart for Hamiltonian variables.

Consider the following functional for a system of N degrees of freedom:

$$S[q_1(t), \ldots, q_N(t)\,;\, p_1(t), \ldots, p_N(t)] = \int_{t_1}^{t_2} \left(\sum_{k=1}^{N} p_k \dot{q}_k - \mathcal{H} \right) dt.$$

You will recognize the integrand as the Lagrangian when expressed in terms of the Hamiltonian $\mathcal{H}$. However, when solving this problem do not assume any properties of $\mathcal{H}$ aside from being some function of the q's and p's. Calculate the first order variation δS when

$$\begin{aligned} q_k(t) &= q_k^\star(t) + \delta q_k(t) \qquad k = 1, \ldots, N \\ p_k(t) &= p_k^\star(t) + \delta p_k(t) \qquad k = 1, \ldots, N, \end{aligned}$$

and show that if it vanishes for *independent* variations of all $2N$ δq's and δp's, then the $2N$ functions $q^\star$ and $p^\star$ satisfy Hamilton's equations. As in the Lagrangian variational principle, it will be necessary for you to assume the variations δq vanish at the endpoints. Is there a corresponding restriction on the variations δp? Is this variational principle still valid when the Hamiltonian has direct time dependence?

Preserving the form of Hamilton's equations

The form of the Euler-Lagrange equations is unchanged when an arbitrary transformation is applied to the generalized coordinates. Is this also true of Hamilton's equations?

Consider the most general time-independent transformation, in the case of one degree of freedom:

$$\begin{aligned} Q &= Q(q, p) \\ P &= P(q, p). \end{aligned}$$

The original and transformed Hamiltonians are related by

$$\mathcal{H}(q, p) = \widetilde{\mathcal{H}}(Q, P).$$

We would like to know, as in the Lagrangian case, whether the equations

$$\dot{Q} = \frac{\partial\widetilde{\mathcal{H}}}{\partial P} \qquad \dot{P} = -\frac{\partial\widetilde{\mathcal{H}}}{\partial Q} \tag{1}$$

are equivalent to

$$\dot{q} = \frac{\partial\mathcal{H}}{\partial p} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q}, \tag{2}$$

for an arbitrary transformation.

Start by expressing $\dot{Q}$ and $\dot{P}$ in (1) in terms of $\dot{q}$ and $\dot{p}$ using the chain rule, and also use the chain rule to express partial derivatives of $\widetilde{\mathcal{H}}$ in terms of partial derivatives of $\mathcal{H}$. You will be able to eliminate the $\dot{p}$ term by multiplying one equation by $\partial P/\partial p$, the other by $\partial Q/\partial p$, and subtracting. The resulting equation will look like the first equation in (2) apart from three factors:

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}$$

$$\frac{\partial q}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial q}{\partial Q}\frac{\partial Q}{\partial p}$$

$$\frac{\partial p}{\partial P}\frac{\partial P}{\partial p} + \frac{\partial p}{\partial Q}\frac{\partial Q}{\partial p}$$

State why (cite properties of partial derivatives) one of these is identically 0, another is identically 1, and the remaining one is the Jacobian of the variable transformation. This shows that not any transformation preserves the form of Hamilton's equations, but only those that have unit Jacobian. In mechanics, transformations with this property are called "canonical".

You will arrive at the same conclusion when eliminating $\dot{q}$ to get the $\dot{p}$ equation, but you don't have to provide those details.

Time evolution is a canonical transformation

Show that the Hamiltonian variable transformation (one degree of freedom) defined by time evolution,

$$\begin{aligned} Q(q(0), p(0)) &= q(\Delta t) \\ P(q(0), p(0)) &= p(\Delta t), \end{aligned}$$

is canonical to first order in Δt.

Poisson bracket notation

Bracketology has a rich tradition in physics, beginning with Poisson. Given arbitrary functions $f(q,p)$ and $g(q,p)$, he introduced the notation

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}.$$

Using this notation, the statement that the transformed variables Q and P are canonical is simply $\{Q,P\} = 1$.

Consider an arbitrary function $A(q,p,t)$. Using the chain rule and something else, show that

$$\dot{A} = \{A,\mathcal{H}\} + \frac{\partial A}{\partial t},$$

where $\mathcal{H}$ is the Hamiltonian. What well-known equations does this give you when $A = q$, $A = p$ or $A = \mathcal{H}$? State a condition, involving the Poisson bracket, such that a function $I(q,p)$ is conserved.

The close correspondence between Poisson bracket equations, and some equations of quantum mechanics featuring the commutator of matrices, served as an inspiration to 20th century bracketologist Dirac.