Assignment 7

Due date: Monday, March 20

**Time-translation symmetry**

Consider a Lagrangian with no direct time dependence,

\[ \mathcal{L}(s) = \mathcal{L}(q_1(s), \ldots, q_N(s); \dot{q}_1(s), \ldots, \dot{q}_N(s)), \]

and where \( s \) parameterizes the continuous transformation

\[ q_k(s; t) = q_k(t + s) \quad k = 1, \ldots, N. \]  

(1)

This transformation is called “time-translation”.

Show that

\[ \left. \frac{d\mathcal{L}(s)}{ds} \right|_{s=0} = \frac{dF}{dt} \]  

for some function \( F \) (that you should determine). This establishes that \( \mathcal{L}(s) \) has time translation invariance.

By Noether’s theorem, Lagrangians with property (2) automatically have the conserved quantity

\[ I = \sum_{k=1}^{N} p_k \left. \frac{dq_k(s)}{ds} \right|_{s=0} - F. \]

For the transformation (1) and the corresponding \( F \) you found above, the quantity \( I \) has another name — what is it?

**Mass-on-a-wheel Hamiltonian**

In this exercise you revisit the mass-on-a-rolling-wheel system of assignment 4. Obtain the Hamiltonian \( \mathcal{H} \) for this system. Sketch contours of constant \( \mathcal{H} \) in phase space and add arrows to indicate the direction of motion. At a particular value \( \mathcal{H} = E^* \) the topology of the phase space orbits changes. Find \( E^* \).
Parallel-transported dipole motion near the equator

In assignment 6 you found the following time evolution equations for a dipole constrained to move over the surface of a sphere by parallel transport:

\[
\cos \theta \ddot{\theta} = \sin \theta \dot{\theta}^2 - \omega_0^2 \cos^2 \theta \cos \alpha
\]

\[
\cos \theta \ddot{\alpha} = - \left( \sin \theta + 2 \frac{\cos^2 \theta}{\sin \theta} \right) \dot{\theta} \dot{\alpha} + \omega_0^2 \frac{\cos^3 \theta}{\sin \theta} \sin \alpha.
\]

Here \( \theta \) is the polar (latitude) angle of the dipole’s position on the sphere and \( \alpha \) is its angle relative to \( \hat{\theta} \) (south). The parameters \( I \) (moment of inertia) and \( \epsilon \) (dipole energy) have been combined to form a frequency: \( \epsilon/I = \omega_0^2 \). We will switch to a dimensionless time defined by \( t' = \omega_0 t \) so that when expressed in terms of \( t' \) even this parameter is absent (the quantity \( \sqrt{I/\epsilon} \) has become our unit of time). The prime on \( t \) is omitted below.

Find approximate solutions for the case where \( \theta(t) = \pi/2 + \beta(t) \) and both \( \beta(t) \) and \( \alpha(t) \) are small. The azimuth (longitude) angle \( \phi \) of the dipole can be recovered from the non-holonomic constraint

\[
\dot{\alpha} + \cos \theta \dot{\phi} = 0
\]

once you know \( \theta(t) \) and \( \alpha(t) \). You will find in the small \( \alpha, \beta \) solutions that approximately

\[
\phi(t) = ct + \phi_0
\]

for some constant \( c \). As a check, the character of the solutions changes when the magnitude of \( c \) exceeds a particular value.