Assignment 7

Due date: Monday, March 20

Time-translation symmetry

Consider a Lagrangian with no direct time dependence,

$$\mathcal{L}(s) = \mathcal{L}(q_1(s), \ldots, q_N(s); \dot{q}_1(s), \ldots, \dot{q}_N(s)),$$

and where s parameterizes the continuous transformation

$$q_k(s;t) = q_k(t+s) \qquad k = 1, \ldots, N. \tag{1}$$

This transformation is called “time-translation”.

Show that

$$\left.\frac{d\mathcal{L}(s)}{ds}\right|_{s=0} = \frac{dF}{dt} \tag{2}$$

for some function F (that you should determine). This establishes that $\mathcal{L}(s)$ has time translation invariance.

By Noether’s theorem, Lagrangians with property (2) automatically have the conserved quantity

$$I = \sum_{k=1}^{N} p_k \left.\frac{dq_k(s)}{ds}\right|_{s=0} - F.$$

For the transformation (1) and the corresponding F you found above, the quantity I has another name — what is it?

Mass-on-a-wheel Hamiltonian

In this exercise you revisit the mass-on-a-rolling-wheel system of assignment 4. Obtain the Hamiltonian $\mathcal{H}$ for this system. Sketch contours of constant $\mathcal{H}$ in phase space and add arrows to indicate the direction of motion. At a particular value $\mathcal{H} = E^*$ the topology of the phase space orbits changes. Find E^*.

Parallel-transported dipole motion near the equator

In assignment 6 you found the following time evolution equations for a dipole constrained to move over the surface of a sphere by parallel transport:

$$\cos\theta\,\ddot{\theta} = \sin\theta\,\dot{\alpha}^2 - \omega_0^2\cos^2\theta\cos\alpha$$
$$\cos\theta\,\ddot{\alpha} = -\left(\sin\theta + 2\frac{\cos^2\theta}{\sin\theta}\right)\dot{\theta}\dot{\alpha} + \omega_0^2\frac{\cos^3\theta}{\sin\theta}\sin\alpha.$$

Here θ is the polar (latitude) angle of the dipole's position on the sphere and α is its angle relative to $\hat{\theta}$ (south). The parameters I (moment of inertia) and ϵ (dipole energy) have been combined to form a frequency: $\epsilon/I = \omega_0^2$. We will switch to a dimensionless time defined by $t' = \omega_0 t$ so that when expressed in terms of t' even this parameter is absent (the quantity $\sqrt{I/\epsilon}$ has become our unit of time). The prime on t is omitted below.

Find approximate solutions for the case where $\theta(t) = \pi/2 + \beta(t)$ and both $\beta(t)$ and $\alpha(t)$ are small. The azimuth (longitude) angle ϕ of the dipole can be recovered from the non-holonomic constraint

$$\dot{\alpha} + \cos\theta\,\dot{\phi} = 0$$

once you know $\theta(t)$ and $\alpha(t)$. You will find in the small α, β solutions that approximately

$$\phi(t) = ct + \phi_0$$

for some constant c. As a check, the character of the solutions changes when the magnitude of c exceeds a particular value.