Assignment 6

Due date: Monday, March 13

Polymer model

Derive the equations of motion for the polymer chain model described in lecture, but for the case of $N = 3$ mass points. Use the method of Lagrange multipliers and, by solving for them explicitly, eliminate them from the equations of motion. **You do not have to solve the equations of motion**.

For the $N = 2$ case analyzed in lecture, the single Lagrange multiplier turned out to be constant in time. Is this also true for the two Lagrange multipliers in this problem?

Although solving for the λ's will seem like a tedious exercise, and increasingly cumbersome for $N > 3$, the fact remains that these equations are linear in the λ's and lend themselves to efficient numerical (as opposed to algebraic) methods. In a large scale simulation, say with $N = 1000$, the computer code would use a linear equation solver to numerically calculate the λ's in each time step.

Parallel-transported dipole

A point magnetic dipole of mass M moves frictionlessly on the surface of a sphere of radius r. The kinetic energy of this two-degree-of-freedom system[1] is

$$T = \frac{1}{2} I \left(\dot\theta^2 + \sin^2\theta \, \dot\phi^2 \right),$$

where θ and ϕ are the standard angles on the sphere and $I = Mr^2$. The dipole interacts with a static magnetic field — similar to the Earth's — produced by a dipole source at the center of the sphere and aligned with the poles in our spherical coordinate system. The resulting potential energy of the interaction is

$$V = \epsilon \sin\theta \cos\alpha,$$

where ϵ is the product of the source magnetic field at the equator ($\theta = \pi/2$) and the magnetic moment of the point dipole, and where the latter has direction

$$\cos\alpha \, \hat\theta + \sin\alpha \, \hat\phi$$

[1] This was a class exercise in an early lecture.

relative to the standard basis $\hat{\theta}$ (south), $\hat{\phi}$ (east). This system presents a new challenge in that α is **not a degree of freedom** but is subject to the parallel-transport-of-vectors constraint that in our coordinates is expressed by

$$\dot{\alpha} + \cos\theta\,\dot{\phi} = 0. \tag{1}$$

Only when this condition is satisfied will the dipole move over the surface of the sphere so that (as seen locally) it never rotates about the normal to the sphere.

(a) Consider an orbit where $\theta(t) = \theta_0$ and $\phi(t)$ winds a complete circle around the sphere. Show[2] that the solid angle Ω enclosed by the orbit in the northern hemisphere and the net change $\Delta\alpha$ are related by

$$\Delta\alpha + 2\pi = \Omega.$$

(b) The fact made plain by the simple closed orbit above — that one cannot solve consistently for α in terms of θ and ϕ — shows that constraint (1) is non-holonomic. Starting with θ, ϕ and α as degrees of freedom, use the method of Lagrange multipliers for non-holonomic constraints such as (1) to solve for the motion. You should end up with two differential equations and one algebraic equation, and two of these will involve a Lagrange multiplier function. Eliminate the Lagrange multiplier and also ϕ, using (1), to arrive at a pair of second order differential equations for θ and α (which you do not have to solve).

Newtonian solution of the "non-holonomic wheel"

This problem revisits the wheel with two degrees of freedom from lecture 10: our introduction to non-holonomic constraints. Here we derive the equations of motion by ordinary Newtonian methods as a check of the equations obtained in lecture using the variational calculus.

The wheel has radius r, mass M, and moment of inertia I_3 about its axis. It rolls without slipping on a plane that is tipped by angle α relative to the horizontal. We will use (x, y) coordinates on the tilted plane, where y increases in the uphill direction. The axis of the wheel makes angle θ with respect to the x axis (so that for $0 < \theta < \pi/2$ the wheel moves toward positive x when rolling downhill). The angle of rotation about the axis, ϕ, is defined so that it increases when the wheel rolls downhill and $\theta = 0$.

(a) By considering torques about the axis through the wheel that is normal to the plane, argue that

$$\theta(t) = \omega t + \theta_0, \tag{2}$$

[2]Use Archimedes' hat-box theorem.

where ω and θ_0 are constants.

(b) Let y' be the axis in the (x, y) plane that is instantaneously parallel to the plane of the wheel (and coincides with y when $\theta = 0$). Three forces, gravity, a normal force, and a static friction force (rolling without slipping) combine to produce a net force F' on the wheel that is purely along the y' axis. Show that

$$F' = F_s - Mg \sin\alpha \cos\theta,$$

where F_s is the component of the static friction force acting along y'.

(c) Starting with (lecture 10)

$$\begin{aligned} \dot{x} &= r\dot{\phi}\sin\theta \\ \dot{y} &= -r\dot{\phi}\cos\theta, \end{aligned} \tag{3}$$

and the formula

$$\ddot{y}' = \cos\theta\, \ddot{y} - \sin\theta\, \ddot{x},$$

for the acceleration along the y' axis, show that

$$F' = M\ddot{y}' = -Mr\ddot{\phi}. \tag{4}$$

(d) Compute the torque about the axis of the wheel and show that

$$I_3\ddot{\phi} = rF_s. \tag{5}$$

(e) Eliminate F_s from equations (4) and (5) and substitute (2) to obtain

$$(I + Mr^2)\ddot{\phi} = rMg \sin\alpha \cos(\omega t + \theta_0).$$

Integrating this equation with respect to t to obtain $\dot{\phi}$, substituting the result into (3), and another integration produces $x(t)$ and $y(t)$ (you do not need to do this last part).