Assignment 6

Due date: Friday, March 18

**Motion with a vector field constraint**

A point mass $m$ is constrained to move in the plane so that its velocity is always perpendicular to a (static) vector field:

\[ \mathbf{c}(x, y) \cdot \mathbf{v} = 0, \]

\[ \mathbf{c}(x, y) = c_x(x, y) \mathbf{\hat{x}} + c_y(x, y) \mathbf{\hat{y}}. \]

In variational language this means that the variations at time $t$, $\delta x(t)$ and $\delta y(t)$, satisfy the linear constraint

\[ c_x \delta x(t) + c_y \delta y(t) = 0. \]

This is a non-holonomic constraint and the Euler-Lagrange equations take the form (as in the rolling wheel system from lecture):

\[ \frac{\delta S}{\delta x(t)} = \lambda(t)c_x \] \hspace{1cm} (2)

\[ \frac{\delta S}{\delta y(t)} = \lambda(t)c_y, \] \hspace{1cm} (3)

where $S$ is the action without the constraint (time integral of the kinetic energy of a mass moving in the plane).

Your assignment is to derive equations of motion for $x(t)$ and $y(t)$ that do not involve the unknown Lagrange multiplier $\lambda(t)$. Start by writing out (2) and (3), and substituting $\ddot{x}$ and $\ddot{y}$ from these into the expression you get by taking one time derivative of (1). This will allow you to solve for $\lambda(t)$, which you can then substitute into (2) and (3).

**Time-translation symmetry**

Consider a Lagrangian with no direct time dependence,

\[ L(s) = L(q_1(s), \ldots, q_N(s); \dot{q}_1(s), \ldots, \dot{q}_N(s)), \]

and where $s$ parameterizes the continuous transformation

\[ q_k(s; t) = q_k(t + s) \quad k = 1, \ldots, N. \] \hspace{1cm} (4)
This transformation is called “time-translation”.

Show that

\[
\frac{dL(s)}{ds}\bigg|_{s=0} = \frac{dF}{dt}
\]

for some function \(F\); find \(F\). This establishes that \(L(s)\) has time translation invariance.

By Noether’s theorem, Lagrangians with property (5) automatically have the conserved quantity

\[
I = \sum_{k=1}^{N} p_k \frac{dq_k(s)}{ds}\bigg|_{s=0} - F.
\]

For the transformation (4) and the corresponding \(F\) you found above, the quantity \(I\) has another name — what is it?

**Hypocycloid Hamiltonian**

In this exercise you revisit the hypocycloid system of assignment 3. The plane of the hypocycloid is now vertical, so in addition to kinetic energy, the mass \(m\) on the rim of the wheel also has gravitational potential energy \(V(\theta) = -mgx(\theta)\). The lowest potential energy occurs at \(\theta = 0\), where \(x(0) = R\).

To keep things simple, fix the ratio of the two radii at \(R/r = 2\). The motion of the mass is very simple in this case — why? [Hint: Look at the formula for the \(y\) coordinate.]

Obtain the Hamiltonian \(H\) for this system.

Sketch contours of constant \(H\) in phase space and add arrows to indicate the direction of motion. Why are you able to restrict the range of \(\theta\) to be between \(-\pi\) and \(+\pi\)? Would this be true for other \(R/r\) ratios?

At a particular value \(H = E^*\) the topology of the phase space orbits changes. Find \(E^*\).