Assignment 5

Due date: Monday, March 6

Time-independent holonomic constraints and $\mathcal{H} = E$

Show that when a mechanical system has time-independent holonomic constraints, so that all the particle positions can be expressed as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_N), \tag{1}$$

then $\mathcal{H} = T + V$. You will need to use (i) the definition of $\mathcal{H}$ in terms of $\mathcal{L} = T - V$, (ii) the general expression for the kinetic energy,

$$T = \frac{1}{2}\sum_i m_i\, \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i, \tag{2}$$

and (iii) the fact that V is independent of the generalized velocities.

Brachistochrone

Starting from the transit-time functional for the brachistochrone problem

$$T[y] = \int_0^l \sqrt{\frac{1 + (dy/dx)^2}{-2gy}}\,dx,$$

whose integrand is independent of x, use the same trick introduced for the hanging chain to derive a first-order differential equation satisfied by $y(x)$.

Next, show that your equation is satisfied by the following parametric representation of the curve,

$$\begin{aligned} x &= R\theta - R\sin\theta \\ y &= -R + R\cos\theta, \end{aligned}$$

where R is a parameter. Sketch this curve as θ ranges from 0 to 2π. What should be the value of R so the curve spans a distance l between the two points where it intercepts the horizontal plane ($y = 0$)?

Finally, plug your parametric solution into the transit-time functional, integrate, and obtain the time of transit. Compare your answer with the bound on the transit time we found in the first lecture (vertical drop, horizontal coast, vertical ascent).

Brachistochrone Jr.

Consider this variation on the brachistochrone problem:

A particle starts at a distance l from a vertical wall. What path should it take, when acted upon by gravity, so that it arrives at the wall in the minimum time?

Solve this problem with a **symmetry argument**. This is a technique not normally taught in textbooks, as there are no general recipes. What you are trying to do in this case is to relate the wall-version of the brachistochrone problem to the standard version without the wall (and thereby avoid having to write down lots of equations). This is a fun problem to discuss with others. The challenge is not just to get the right answer (for the transit time), but a tight argument.

Extrema of the harmonic oscillator action

The action functional for a 1D harmonic oscillator is

$$S[x] = \int_0^T \left(\tfrac{1}{2}m\,\dot{x}^2 - \tfrac{1}{2}m\omega_0^2\,x^2\right) dt,$$

and the trajectory endpoints are fixed as

$$x(0) = x_1, \qquad x(T) = x_2.$$

In this problem you will study arbitrary trajectories when expressed in the form

$$x(t) = x^\star(t) + \delta x(t),$$

where $x^\star(t)$ is an extremal trajectory given by Hamilton's principle (and therefore satisfies the Euler-Lagrange equation) and $\delta x(t)$ is whatever is left over and is **not assumed to be small**.

(a) Show that $S[x] = S[x^\star] + \delta S$, where

$$\delta S = \int_0^T \left(\tfrac{1}{2}m\,\delta\dot{x}^2 - \tfrac{1}{2}m\omega_0^2\,\delta x^2\right) dt.$$

Since $\delta x(0) = \delta x(T) = 0$, consider perturbations having the form

$$\delta x(t) = \Delta \sin\left(N\pi t/T\right),$$

where Δ is the amplitude of the perturbation and the integer N counts the number of wiggles between the endpoints.

(b) Using the above form for $\delta x(t)$, show that $\delta S = c_N \Delta^2$, and determine the constant c_N. Further, show that for the case $\omega_0 T > \pi$ (trajectories that span more than one half-period of oscillation), c_N can have either sign, depending on N. The action functional thus will not always be a simple minimum or maximum at the extremum, but more generally, a "saddle" having both signs of curvature, depending on the "direction" in the space of perturbations.