Assignment 4

Due date: Monday, February 27

Swapped derivatives

This exercise sheds some light on the mysteries of "partial" and "total" derivatives when they occur in combinations. We will consider functions $f(q,\dot{q},t)$, that is, functions of three arguments where the first two have a special relationship. This relationship is irrelevant when applying partial derivatives. The following notation may help:

$$\frac{\partial f}{\partial q} = f_1,$$

where f_1 is the function you get when you take the derivative of f with respect to the first argument. Similarly,

$$\frac{\partial f}{\partial \dot{q}} = f_2,$$

and multiple derivatives are written as

$$\frac{\partial^2 f}{\partial q \partial q} = f_{11} \qquad \frac{\partial^2 f}{\partial q \partial \dot{q}} = f_{12} \qquad \text{etc.}$$

The total derivative d/dt brings out the relationship between q and $\dot{q}$ (and even $\ddot{q}$). When applying it to f we use the chain rule:

$$\frac{df}{dt} = f_1\dot{q} + f_2\ddot{q} + f_3.$$

Now it's your turn. First show that

$$\frac{d}{dt}\frac{\partial f}{\partial q} = \frac{\partial}{\partial q}\frac{df}{dt}\ .$$

We can restate this as

$$\frac{\partial}{\partial q}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial q} = 0$$

with the understanding that the derivatives always act on functions of the form $f(q,\dot{q},t)$. Next, show that when replacing q with $\dot{q}$ we get a different rule:

$$\frac{\partial}{\partial \dot{q}}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}} = \frac{\partial}{\partial q}\ .$$

Finally, use the fact that the Lagrangian is exactly the type of function we have here to rewrite the Euler-Lagrange equations in the form

$$0 = \frac{\partial \dot{L}}{\partial \dot{q}_k} - \cdots$$

where you are supposed to work out the rest of the equation.

Mass on a rolling wheel

A massless wheel of radius R rolls without slipping along the x-axis. There is a mass m fixed to the wheel, at a distance $r < R$ from the hub on which gravity acts as the wheel rolls, always in a vertical plane. Use the angle θ the wheel has rolled for your generalized coordinate. When $\theta = 0$ the mass is at its lowest point and the rim touches the x-axis at $x = 0$; positive θ corresponds to positive x.

Write out the Lagrangian for this system and from that determine the equation of motion for θ. Obtain an approximate equation of motion valid when θ is always small, *i.e.* when the energy is slightly above the minimum possible energy and the wheel oscillates about $\theta = 0$. What is the frequency of small oscillations?

Rigid body simulation 2.0

Redo the rigid body simulation exercise of assignment 3, but get it right! Decrease the two small parameters so that you trust your answer for the rotation angle θ (in radians) to four digits.