Assignment 4

Due date: Wednesday, March 2

*Time-independent holonomic constraints and $H = E*$

Show that when a mechanical system has time-independent holonomic constraints, so that all the particle positions can be expressed as

$$r_i = r_i(q_1, \ldots, q_N),$$  \hspace{1cm} (1)

then $H = T + V$. You will need to use (i) the definition of $H$ in terms of $L = T - V$, (i) the general expression for the kinetic energy,

$$T = \frac{1}{2} \sum_i m_i \dot{r}_i \cdot \dot{r}_i,$$  \hspace{1cm} (2)

and (iii) the fact that $V$ is independent of the generalized velocities.

*Brachistochrone*

Starting from the transit-time functional for the brachistochrone problem

$$\Delta t[y] = \int_0^l \sqrt{\frac{1 + (dy/dx)^2}{-2gy}} \, dx,$$

derive the Euler-Lagrange equation for the minimum-time curve $y(x)$.

Next, show that your equation is satisfied by the following parametric representation of the curve,

$$x = R \theta - R \sin \theta$$
$$y = -R + R \cos \theta,$$

where $R$ is a parameter. Sketch this curve as $\theta$ ranges from 0 to $2\pi$. What should be the value of $R$ so the curve spans a distance $l$ between the two points where it intercepts the horizontal plane ($y = 0$)?

Finally, plug your parametric solution into the transit-time functional, integrate, and obtain the time of transit. Compare your answer with the bound on the transit time we found in the first lecture.
Brachistochrone Jr.

Consider this variation on the brachistochrone problem:

A particle starts at a distance \( l \) from a vertical wall. What path should it take, when acted upon by gravity, so that it arrives at the wall in the minimum time?

Solve this problem with a symmetry argument. This is a technique not normally taught in textbooks, as there are no general recipes. What you are trying to do in this case is to relate the wall-version of the brachistochrone problem to the standard version without the wall (and thereby avoid having to write down lots of equations). This is a fun problem to discuss with others. The challenge is not just to get the right answer (for the transit time), but a tight argument.