Assignment 2

Due date: Monday, February 13

Gyrating ring, part II

Revisit the gyrating ring of the first assignment. Again we are only interested in the special type of motion where the ring's center is fixed and the point of contact with the table moves clockwise on a circle with angular velocity $\Omega = 2\pi/T$.

First show that the ring is *not* freely-precessing, because its angular momentum is not constant. Recall that the direction of the angular velocity has a special orientation with respect to the ring. Use this fact to argue that the angular momentum satisfies $\mathbf{L} = c\,\boldsymbol{\omega}$ where c is a simple scalar constant (which you should specify). Thus $\mathbf{L}$, like $\boldsymbol{\omega}$, precesses about the table normal with angular velocity Ω.

From the time derivative of $\mathbf{L}$ obtain a vector formula for the torque $\mathbf{N}$ on the ring in terms of $\boldsymbol{\omega}$, the table normal $\hat{\mathbf{z}}$, and scalar constants.

By equating the magnitude of the $\mathbf{N}$ you found above with the torque about the ring center due to the table contact force, show that Ω is not a free parameter but determined by the tilt angle α (and some constants).

Obtain a formula for the total energy (kinetic and gravitational) E of the gyrating ring, where $E = 0$ corresponds to the ring at rest and lying flat on the table ($\alpha = 0$). Use this formula to express Ω in terms of E (and constants) instead of α.

Finally, suppose friction decreases E linearly to zero with time. What would be the resulting time dependence of Ω?

Angular momentum of a rigid body

Modify the derivation used in lecture 4 for the kinetic energy of a rigid body in order to obtain its angular momentum $\mathbf{L}$. First show that $\mathbf{L}$ is comprised of two parts, one of which only involves the body's center of mass $\mathbf{R}$, and the other just the positions $\mathbf{r}_i$ of the masses relative to $\mathbf{R}$.

Now suppose $\mathbf{R}$ is at rest, so $\mathbf{L}$ is entirely due to the rotational motion of the body. Show that $\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$, where $\mathbf{I}$ is the same moment of inertia tensor that appeared in the derivation of the rotational kinetic energy.

Time derivative of the moment of inertia tensor

Use the principal axis form of the moment of inertia tensor

$$\mathbf{I} = I_1\,\hat{\mathbf{1}}\hat{\mathbf{1}} + I_2\,\hat{\mathbf{2}}\hat{\mathbf{2}} + I_3\,\hat{\mathbf{3}}\hat{\mathbf{3}}, \tag{1}$$

and the fact that the body-fixed principal moments of inertia are time indepenedent, to show that

$$\dot{\mathbf{I}} = \boldsymbol{\omega} \times \mathbf{I} - \mathbf{I} \times \boldsymbol{\omega}. \tag{2}$$

Next, with $\mathbf{I}$ expressed in the principal basis (1) as well as

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \omega_2\,\hat{\mathbf{2}} + \omega_3\,\hat{\mathbf{3}},$$

write out the right hand side of (2) only in terms of the principal basis vectors. Use this expression to establish the following conditions for $\mathbf{I}$ to be constant.

- When $I_1 < I_2 < I_3$, then $\mathbf{I}$ is constant only when the angular velocity is exactly zero.
- When $I_1 = I_2 = I_3$, then $\mathbf{I}$ is constant for *any* angular velocity.
- Finally, in the case of a symmetric top, $I_1 = I_2 \neq I_3$, $\mathbf{I}$ is constant when the angular velocity has a particular form (which you should specify).

Principal moments of a solid cylinder

Calculate the principal moments of inertia of a solid cylinder of uniform mass density ρ, radius r and height h. Exploit the symmetry of the body in your calculation so the components of $\mathbf{I}$ have the principal (diagonal) form. Two of your moments will be equal and different from the third (symmetric top), except at a special value of the ratio h/r (which you should determine) when all are equal. A solid cylinder having this special size ratio will be indistinguishable, in its rotational behavior, from a sphere.