Assignment 13

Due date: Wednesday, May 10

Relativistic string equation of motion

Show that the Euler-Lagrange equations,

\[ 0 = \frac{\partial p}{\partial \left( \frac{\partial L}{\partial \dot{s}^\alpha} \right)} + \frac{\partial q}{\partial \left( \frac{\partial L}{\partial \dot{s}^\alpha} \right)} = \frac{\partial a}{\partial \left( \frac{\partial L}{\partial \dot{s}^\alpha} \right)}, \]

applied to the Lagrangian of the relativistic string produces the equation of motion

\[ 0 = \epsilon^{ab}(\partial_a v^{\alpha\beta})(\partial_b s^\beta), \]

(1)

where

\[ v^{\alpha\beta} = a^{\alpha\beta}/L \]

is a normalized rescaling of the anti-symmetric tangent-space tensor \( a^{\alpha\beta} \), analogous to the 4-velocity we define for the point particle.

Relativistic string as a non-linear modification of simple elastic string

Consider a relativistic string that resides only in the \((x, y)\) plane and has infinite extent in \(x\). The simple (non-relativistic) elastic string, in this geometry, is described by a function \(y(x, t)\) that gives the displacement as a function of time \(t\) at all positions \(x\). Suppose we likewise use \((x, t)\) as the parameters of the relativistic string’s world-surface:

\[ s^\alpha(x, t) = (ct, x, y(x, t), 0). \]

The action of a relativistic string with this parameterization is a functional of \(y(x, t)\):

\[ S[y] = \int L(\partial_x y, \partial_t y) dx dt. \]

Find the Lagrangian function \(L\) and from that show that the resulting (Euler-Lagrange) equation of motion for \(y\) is the wave equation for the simple elastic string with additional cubic terms. Show that arbitrary running waves, \(y = f(x - ct)\) and \(y = g(x + ct)\), are solutions, but that this does not extend to linear combinations of right and left moving waves.

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\(^1\)Einstein summation convention for all repeated indices, greek and latin.
Local conservation law for relativistic string

Show that the stress-energy tensor density for relativistic strings,

\[ T^{\alpha\beta}(x) = \int (\mathcal{L} \, dp \, dq) \, v^{\alpha\gamma} v^{\beta}_{\gamma} \, \delta^4 (x - s(p,q)), \]

is locally conserved, that is,

\[ \partial_\beta T^{\alpha\beta}(x) = 0. \]

Here \( v^{\alpha\beta} \) is the normalized tensor (2). Your derivation should mirror the derivation in lecture of the conservation law for the point-particle stress-energy tensor. In particular, you will need to use the string’s equations of motion (1).