Assignment 13

Due date: Wednesday, May 10

Relativistic string equation of motion

Show that the Euler-Lagrange equations[1],

$$0 = \partial_p \left(\frac{\partial \mathcal{L}}{\partial(\partial_p s^\alpha)} \right) + \partial_q \left(\frac{\partial \mathcal{L}}{\partial(\partial_q s^\alpha)} \right) = \partial_a \left(\frac{\partial \mathcal{L}}{\partial(\partial_a s^\alpha)} \right),$$

applied to the Lagrangian of the relativistic string produces the equation of motion

$$0 = \epsilon^{ab}(\partial_a v^{\alpha\beta})(\partial_b s_\beta), \tag{1}$$

where

$$v^{\alpha\beta} = a^{\alpha\beta}/\mathcal{L} \tag{2}$$

is a normalized rescaling of the anti-symmetric tangent-space tensor $a^{\alpha\beta}$, analogous to the 4-velocity we define for the point particle.

Relativistic string as a non-linear modification of simple elastic string

Consider a relativistic string that resides only in the (x, y) plane and has infinite extent in x. The simple (non-relativistic) elastic string, in this geometry, is described by a function $y(x, t)$ that gives the displacement as a function of time t at all positions x. Suppose we likewise use (x, t) as the parameters of the relativistic string's world-surface:

$$s^\alpha(x, t) = (ct\,, x\,, y(x, t)\,, 0).$$

The action of a relativistic string with this parameterization is a functional of $y(x, t)$:

$$S[y] = \int \mathcal{L}(\partial_x y, \partial_t y) dx\, dt.$$

Find the Lagrangian function $\mathcal{L}$ and from that show that the resulting (Euler-Lagrange) equation of motion for y is the wave equation for the simple elastic string with additional cubic terms. Show that arbitrary running waves, $y = f(x - ct)$ and $y = g(x + ct)$, are solutions, but that this does not extend to linear combinations of right and left moving waves.

[1] Einstein summation convention for all repeated indices, greek and latin.

Local conservation law for relativistic string

Show that the stress-energy tensor density for relativistic strings,

$$T^{\alpha\beta}(x) = \int (\mathcal{L}\, dp\, dq)\, v^{\alpha\gamma} v_{\gamma}{}^{\beta}\, \delta^4 \left(x - s(p,q)\right),$$

is locally conserved, that is,

$$\partial_\beta T^{\alpha\beta}(x) = 0.$$

Here $v^{\alpha\beta}$ is the normalized tensor (2). Your derivation should mirror the derivation in lecture of the conservation law for the point-particle stress-energy tensor. In particular, you will need to use the string's equations of motion (1).