Assignment 12

Due date: Monday, May 1

Energy in terms of orbit parameters

Show that the energy expression derived in lecture,

$$E = \frac{L_z^2}{2\mu}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) - Au,$$

reduces to the simple formula

$$E = (\epsilon^2 - 1)\frac{A}{2r_0}.$$

Kepler's 2nd Law

Derive Kepler's law relating the orbital period T and semi-major axis a,

$$T^2 = \frac{4\pi^2}{G(M_1 + M_2)}a^3,$$

by equating the dynamical,

$$A = \oint \dot{A}\, dt = \frac{L_z}{2\mu}T,$$

and geometric,

$$A = \pi ab,$$

formulas for the orbit area A.

Martian gravity assist

Approximate the orbits of Earth and Mars as circular with radii r_E and r_M, where $r_M/r_E = 3/2$. A space probe is launched from the Earth with velocity parallel to the Earth's (instantaneous) velocity and of a magnitude such that the aphelion of the probe's elliptic orbit equals r_M. What is the eccentricity ϵ of the probe's orbit?

The probe's arrival at the orbit of Mars is timed so that it experiences the maximum gravity assist from Mars (their velocities are nearly parallel when Mars overtakes the probe). What is the eccentricity ϵ' of the probe's orbit after its encounter with Mars? Calculate $r_{\max}/r_M$, where $r_{\max}$ is the aphelion of the new orbit.

Hint: For the second part, first derive the following velocity relationships:

$$\frac{v_{\min}}{v_{\max}} = \frac{r_{\max}}{r_{\min}}, \qquad \left(\frac{v_{\min}}{v_0}\right)^2 = 1 + \epsilon.$$

The first identity compares speeds and distances at perihelion (subscript "min") and aphelion (subscript "max") points of an orbit while the second relates perihelion properties of two orbits with the same perihelion and different eccentricity (v_0 is the speed of the circular orbit). Also, you should treat the sun as fixed in space and approximate μ by the much smaller mass of the orbiting body (planet or probe).