Assignment 11

Due date: Monday, April 24

Perturbed circular orbit and precession

Repeat the calculation in lecture, of the two orbital frequencies ω_θ (orbit completion) and ω_r (radial oscillation) when the force law is slightly different from a pure inverse square. Specifically, take as the potential $U(r)$ — which includes the "centrifugal barrier" — the function

$$U(r) = \frac{B}{r^2} - \frac{A}{r} - \frac{C}{r^3}.$$

As before, $A = GM_1M_2$ and $B = L_z^2/2\mu$. The last term, with strength C, arises when the mass distribution within one of the bodies is non-spherical. For example, a moon or spacecraft in orbit within the equatorial plane of an oblate planet like Jupiter will experience a potential of exactly this form with $C > 0$. It is important that you treat C as a tiny correction. In fact, you should in all your calculations **keep only the lowest order correction caused by** C. Here is a guide to your calculations[1]:

1. Calculate the new, corrected, equilibrium radius

   $$r_1 = r_0 + \cdots ,$$

   where $r_0 = 2B/A$ and $\cdots$ is a single term proportional to C.
2. Calculate the correction to the curvature of U at the new equilibrium point:

   $$K = U''(r_1) = \frac{A}{r_0^3} + \alpha\frac{C}{r_0^5}.$$

   Here we have given you the answer up to a numerical factor α (which you need to find).
3. Calculate the corrections to ω_θ and ω_r (again, keeping only terms proportional to C).
4. Form the ratio

   $$\omega_r/\omega_\theta = 1 + \beta\frac{C}{Br_0}$$

   (yes, you need to find the number β) and from this determine the excess or deficit angle $\delta\theta$ by which the orbit fails to close. In other words, you will have found that the axis of the orbit (defined by radial motion) precesses by $\delta\theta$ over the course of one radial oscillation, *i.e.* from one periapsis to the next.

[1] Show all your work and do not use symbolic computation software such as *Mathematica*.

Synchrony of Jupiter's moons

By timing many orbits of the Jovian moons Io, Europa, and Ganymede one obtains very precise values of their average orbital periods:

$$\begin{aligned} T_I &= 1.769137786 \text{ days} \\ T_E &= 3.551181041 \text{ days} \\ T_G &= 7.15455296 \text{ days.} \end{aligned}$$

These are very close to being in the ratio 1:2:4 — but not quite! Show that the corresponding orbital angular frequencies may be very accurately expressed in terms of just **two** basic frequencies, ω_0 and ω_p:

$$\begin{aligned} \omega_I &= 4\omega_0 + \omega_p \\ \omega_E &= 2\omega_0 + \omega_p \\ \omega_G &= \omega_0 + \omega_p. \end{aligned}$$

What is restored after each period $T_0 = 2\pi/\omega_0$? If one of the moons is most responsible — by the mechanism of the first problem — for the small non-zero ω_p, which would it be? Is the sign of ω_p consistent with that mechanism?

A second magic force law

Show that the two-body attractive potential $V(r) = Ar^2$ ($A > 0$) also produces exactly closed orbits, *i.e.* radial oscillations of the circular orbit have the same period as the orbit itself. Hint: $r^2 = x^2 + y^2$.