Assignment 10

Due date: Monday, April 25

Perturbed circular orbit and precession

Repeat the calculation in lecture, of the two orbital frequencies $\omega_\theta$ (orbit completion) and $\omega_r$ (radial oscillation) when the force law is slightly different from a pure inverse square. Specifically, take as the potential $U(r)$ — which includes the “centrifugal barrier” — the function

$$U(r) = \frac{B}{r^2} - \frac{A}{r} - \frac{C}{r^3}.$$ 

As before, $A = GM_1M_2$ and $B = L_z^2/2\mu$. The last term, with strength $C$, arises when the mass distribution within one of the bodies is non-spherical. For example, a moon or spacecraft in orbit within the equatorial plane of an oblate planet like Jupiter will experience a potential of exactly this form with $C > 0$. It is important that you treat $C$ as a tiny correction. In fact, you should in all your calculations keep only the lowest order correction caused by $C$. Here is a guide to your calculations\(^1\):

1. Calculate the new, corrected, equilibrium radius

$$r_1 = r_0 + \cdots,$$

where $r_0 = 2B/A$ and $\cdots$ is a single term proportional to $C$.

2. Calculate the correction to the curvature of $U$ at the new equilibrium point:

$$K = U''(r_1) = \frac{A}{r_0^3} + \frac{C}{r_0^5}.$$ 

Here we have given you the answer up to a numerical factor $\alpha$ (which you need to find).

3. Calculate the corrections to $\omega_\theta$ and $\omega_r$ (again, keeping only terms proportional to $C$).

4. Form the ratio

$$\frac{\omega_r}{\omega_\theta} = 1 + \beta \frac{C}{Br_0}$$

(Yes, you need to find the number $\beta$) and from this determine the excess or deficit angle $\delta \theta$ by which the orbit fails to close. In other words, you will have found that the axis of the orbit (defined by radial motion) precesses by $\delta \theta$ over the course of one radial oscillation, i.e. from one periapsis to the next.

\(^1\)Show all your work and do not use symbolic computation software such as Mathematica.
Synchrony of Jupiter’s moons

By timing many orbits of the Jovian moons Io, Europa, and Ganymede one obtains very precise values of their average orbital periods:

\[
T_I = 1.769137786 \text{ days} \\
T_E = 3.551181041 \text{ days} \\
T_G = 7.15455296 \text{ days}.
\]

These are very close to being in the ratio 1:2:4 — but not quite! Show that the corresponding orbital angular frequencies may be very accurately expressed in terms of just two basic frequencies, \(\omega_0\) and \(\omega_p\):

\[
\omega_I = 4\omega_0 + \omega_p \\
\omega_E = 2\omega_0 + \omega_p \\
\omega_G = \omega_0 + \omega_p.
\]

What is restored after each period \(T_0 = 2\pi/\omega_0\)? If one of the moons is most responsible — by the mechanism of first problem — for the small non-zero \(\omega_p\), which would it be? Is the sign of \(\omega_p\) consistent with that mechanism?

A second magic force law

Show that the two-body attractive potential \(V(r) = Ar^2 \ (A > 0)\) also produces exactly closed orbits, i.e. radial oscillations of the circular orbit have the same period as the orbit itself. Hint: \(r^2 = x^2 + y^2\).