Assignment 10

Due date: Monday, April 17

Ergodicity and the position distribution in a general potential

In the previous homework you found that the distribution of positions (x, y) of the simple Yang-Mills oscillator — an ergodic system — was uniform over the accessible region in the x-y plane. Is this true in general?

Consider a particle moving in D dimensions with Hamiltonian

$$\mathcal{H}(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2M} + V(\mathbf{x}).$$

When the energy E_0 of the particle is precisely defined — something that is at odds with what is allowed in quantum mechanics — the motion is confined to a surface in phase space having one less than the $2D$ dimensions we need for phase space volumes. To avoid this problem we let the energy range within an interval ΔE centered on E_0, where ΔE is as small as we wish.

Your task is to compare the rates the particle visits different positions $\mathbf{x}$ assuming the ergodic hypothesis holds, *i.e.* all accessible phase space subvolumes are visited at the same rate.

Start with this formula for the volume of the entire accessible phase space,

$$\mathrm{vol}(\Delta E) = \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^D\mathbf{x} \int d^D\mathbf{p}\ \delta\left(\mathcal{H}(\mathbf{x}, \mathbf{p}) - E\right)$$

and take the following steps:

1. Because the integrand only depends on the magnitude $p = |\mathbf{p}|$ of the momentum, the D-dimensional momentum integral reduces to a 1-dimensional integral:
$$\int d^D\mathbf{p}\ \cdots = \Omega_D \int p^{D-1} dp\ \cdots .$$
Here Ω_D is the surface area of a unit sphere in D dimensions (whose numerical value we do not need to know).
2. Introduce a change of variable in the momentum integral from p to the variable
$$E' = \mathcal{H}(\mathbf{x}, \mathbf{p}) - E = \frac{p^2}{2M} + V(\mathbf{x}) - E,$$

do the E' integral (integrate the delta function) and obtain

$$\mathrm{vol}(\Delta E) \propto \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \int d^D\mathbf{x}\, |E - V(\mathbf{x})|^{(D-2)/2},$$

where the proportionality hides constant factors such as M and Ω_D.

3. Do the integral over the small energy range to obtain

$$\mathrm{vol}(\Delta E) \propto \int d^D\mathbf{x}\, \rho(\mathbf{x}),$$

and a formula for the distribution of position, $\rho(\mathbf{x})$. As you see, the case of two dimensions is special. How do you rationalize the counter-intuitive fact that in dimensions $D > 2$ the particle spends *more* time in regions of high kinetic energy?

Eliminating time-dependence order by order

In lecture we prove adiabatic invariance of the simple pendulum by applying a sequence of time-dependent canonical transformations that minimize time dependence in the Hamiltonian.

We start with the Hamiltonian

$$\mathcal{H}(\theta, I, t) = (\omega(\epsilon t) + \epsilon h(\theta, \epsilon t))\, I,$$

where ϵ is a fixed small parameter and $\omega(s)$, $h(\theta, s)$ are functions whose form you do not need to know, only that h has periodicity 2π in the first argument (in lecture these are derived for the pendulum). Use the time-dependent generating function

$$F(\theta, I', t) = \left(\theta - \frac{\epsilon}{\omega(\epsilon t)} \int_0^\theta h(\tilde{\theta}, \epsilon t) d\tilde{\theta}\right) I'$$

to transform to the Hamiltonian $\mathcal{H}'(\theta', I', t)$. Note that this is a generating function of the type F_3 (lecture 24), where the second argument is the transformed conjugate momentum. Show that

$$\mathcal{H}'(\theta', I', t) = \left(\omega(\epsilon t) + \epsilon^2 h'(\theta', \epsilon t) + O(\epsilon^3)\right) I',$$

and express the new function h' in terms of h and $\partial h/\partial s$. As you can see, by performing a sequence of such transformations one can supress the time dependence from the angle variable to arbitrary order in ϵ.

Adiabatic switching functions

The proof in lecture of adiabatic invariance of the pendulum action required that the string length is switched between $l(0) = l_1$ and $l(1) = l_2$ by an infinitely smooth function. Of the infinite variety of such functions, consider the following explicit example of an "adiabatic switching function":

$$l(s) = \frac{1}{2}(l_1 + l_2) + \frac{f(s)}{2}(l_2 - l_1), \qquad 0 < s < 1,$$

where

$$f(s) = \tanh\left(\frac{s - 1/2}{s(1-s)}\right).$$

Sketch the function $l(s)$.

Recall that when transformed to angle-action variables, the pendulum Hamiltonian has the time-dependent form

$$\begin{aligned} \mathcal{H}(\theta, I, t) &= \omega I + \epsilon \frac{3}{2} I \left(\frac{1}{l}\frac{dl}{ds}\right) \sin\theta\cos\theta \\ &= \left(\omega(\epsilon t) + \epsilon h(\theta, \epsilon t)\right) I. \end{aligned}$$

For the adiabatic switching function $l(s)$ defined above, show that $h(\theta, s)$ and all its derivatives $\partial^n h/\partial s^n$ vanish for $s \to 0$ and $s \to 1$.

By some extra work (not part of this assignment) one can show that the vanishing of h and its s-derivatives at the endpoints carries over to the function h' in the transformed Hamiltonian $\mathcal{H}'(\theta', I', t)$ with coefficient ϵ^2 (previous problem).