Assignment 1

Due date: Friday, February 3

Gyrating ring

A ring of radius $R$ rolls without slipping on a table while its axis is inclined with respect to the vertical by angle $\alpha$. As the ring gyrates, its center of mass stays fixed in space and its point of contact with the table moves around a circle with period $T$. Obtain an explicit expression for the ring’s angular velocity vector $\omega$.

Rolling sphere

A sphere of radius $R$ rolls without slipping on a table. Interpret “rolling without slipping” in this case as the property that the point of the sphere making contact with the table is instantaneously at rest. Obtain a vector relationship between the sphere’s angular velocity $\omega$, linear velocity $v$, and the unit normal vector of the table, $n$.

The number of “degrees of freedom” is usually defined as the number of continuous parameters required to uniquely specify the positions in a mechanical system. A better definition counts the number of independent velocity components of the system’s motion. According to the better definition, how many degrees of freedom does a rolling ball have?

Time dependent angular velocity

The body frame (of some body) is related to the space frame by the orthogonal matrix $U$. We learned that the time rate of change of $U$ satisfies the equation $\dot{U} = AU$, where

$$
A = \begin{bmatrix}
0 & -\omega_z & \omega_y \\
\omega_z & 0 & -\omega_x \\
-\omega_y & \omega_x & 0
\end{bmatrix}
$$

is an antisymmetric matrix whose nonzero elements correspond to the components of the angular velocity vector in the space frame. Consider a situation, as in the gyrating ring problem, where $\omega$ has the following time-dependence:

$$
\omega_x = (\omega \cos \alpha) \cos \Omega t \quad \omega_y = (\omega \cos \alpha) \sin \Omega t \quad \omega_z = \omega \sin \alpha.
$$

Suppose the space and body frames coincide at $t = 0$, so $U(0)$ is the identity matrix. Take a period of time $T = 2\pi / \Omega$, so $\omega$ completes one period. The rotation matrix $U(T)$ is now the net rotation of the body after the angular velocity has completed one orbit.
Write a simple computer program to compute $U(T)$ by implementing a finite-difference integration of the equation $\dot{U} = AU$:

$$U(t + \Delta t) - U(t) = \Delta t A(t) U(t).$$

- Do not use a canned differential equation solver; program this from scratch as a finite-difference iteration.
- Check that your final $U(T)$ is nearly orthogonal; if not, you need to decrease $\Delta t$.
- Output the resulting $U(T)$ for the case of the gyrating ring (where $\omega$ and $\Omega$ have a special relationship) with $\alpha = \pi/4$. Check your answer by calculating $U(T)$ directly, by comparing the circumference of the ring with the circumference of the ring of contact on the table (along which there is no slipping).