Bessel function notes

In case you had trouble making out the equations on the videos, here they are again. The functions in theses notes that bear the dimension D as a subscript are natural generalizations of the usual Bessel functions that are specific to $D = 2$. The definitions are based on angular averaging applied to plane waves. These notes only cover *uniform* angular averaging. There is a further generalization, to averages involving D-dimensional spherical harmonics, that is also useful occasionally.

The functions $\mathcal{J}_D$ are simple angular averages of plane waves:

$$\mathcal{J}_D(r, q) = \langle e^{i\mathbf{q}\cdot\mathbf{x}} \rangle_{\hat{\mathbf{q}}} = \langle e^{i\mathbf{q}\cdot\mathbf{x}} \rangle_{\hat{\mathbf{x}}}, \qquad r = |\mathbf{x}|,\ q = |\mathbf{q}|.$$

Even though these are always functions of the product qr, we use the two argument notation for consistency with the other Bessel functions (which need not be functions of qr). All the Bessel functions have a useful normalization property and satisfy a differential equation. Here they are for $\mathcal{J}_D$:

$$\mathcal{J}_D(0, q) = \mathcal{J}_D(r, 0) = 1,$$

$$(-\nabla^2 - q^2)\mathcal{J}_D = 0.$$

In these notes ∇^2 always acts on position and is therefore just the radial part of the Laplacian

$$\nabla^2 = \frac{1}{r^{D-1}}\partial_r \left(r^{D-1}\partial_r \right)$$

when acting on the Bessel functions. Here are the first three $\mathcal{J}$ functions:

D	$\mathcal{J}_D(r, q)$
1	$\cos qr$
2	$J_0(qr)$
3	$\sin qr/qr$

The $\mathcal{K}$ family of functions is defined by an integral which averages plane waves not just by angle but also magnitude:

$$\mathcal{K}_D(r,q) = A_D \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+q^2}.$$

For these functions we use a Gauss's law motivated normalization convention, involving the surface area A_D of the unit sphere in D dimensions. To see this, first note that the integral satisfies the following differential equation:

$$(-\nabla^2 + q^2)\mathcal{K}_D = A_D\,\delta^D(\mathbf{x}).$$

Apart from the q^2 term, this is the Poisson equation with a point source at the origin. Near the origin, where the solution diverges, the Laplacian term dominates the q^2 term and the two equations have the same behavior. The nicest way to characterize the behavior is through the flux of the derivative (electric field) through a small sphere enclosing the origin (which also involves A_D):

$$\partial_r \mathcal{K}_D(r,q) \sim \frac{-1}{r^{D-1}}, \qquad r \to 0.$$

Here are the first three $\mathcal{K}$ functions:

D	$\mathcal{K}_D(r,q)$
1	$\exp(-qr)/q$
2	$K_0(qr)$
3	$\exp(-qr)/r$

The third family of Bessel functions (with symbol recognizing Hankel) is defined by an integral similar to the $\mathcal{K}$ family:

$$\mathcal{H}_D(r,q) = \lim_{\epsilon\to 0^+} A_D \int \frac{d^D k}{(2\pi)^D} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2 - q^2 - i\epsilon}.$$

In addition to the reversal in the sign of the q^2 term, a new feature is the infinitesimal ϵ in the denominator, without which the integral is not well defined. The differential equation satisfied by the $\mathcal{H}$ functions (in the $\epsilon \to 0^+$ limit) is the same as for the $\mathcal{K}$ functions but with a change in sign of the q^2 term:

$$(-\nabla^2 - q^2)\mathcal{H}_D = A_D\, \delta^D(\mathbf{x}).$$

Also, because only the source and not the q^2 term determines the divergence at the origin, we have exactly the same behavior there:

$$\partial_r \mathcal{H}_D(r,q) \sim \frac{-1}{r^{D-1}}, \qquad r \to 0.$$

What the ϵ term *does* affect is the large r asymptotic behavior. In dimensions $D = 1$ and $D = 3$, where the integral can be evaluated explicitly in terms of elementary functions, this manifests itself in the selection of the pole when performing a contour integral. Here they are tabulated, along with the case $D = 2$ for which there is no elementary function:

D	$\mathcal{H}_D(r,q)$
1	$i\exp(iqr)/q$
2	$i(\pi/2)H_0(qr)$
3	$\exp(iqr)/r$

It is unfortunate that the standard Hankel function definition for $D = 2$ — unlike J_0 and K_0 — is off by the factor $i(\pi/2)$. But normalization conventions aside, what these functions have in common is a phase that advances with *increasing* r. The complex conjugates of these functions (equivalent to reversing the sign of ϵ in the definition) advance their phase with decreasing r. How the phase should advance comes up when solving the wave equation with a source. Say you are solving the scalar wave equation

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(\mathbf{x},t) = A_D\, \delta^D(\mathbf{x}).$$

From the differential equation above for $\mathcal{H}_D$ you can see that this is solved by the function

$$\Psi(\mathbf{x},t) = e^{-i\omega t}\, \mathcal{H}_D(|\mathbf{x}|, \omega/c),$$

where $\omega > 0$ is the frequency of the wave. Normally when we have a source in a wave equation there is the understanding that all solutions — in particular wave packets — have the form of outgoing waves. It is for this reason that we use the phase-advance-with-increasing-r choice of $\mathcal{H}$ function when our positive frequency time dependence convention is $e^{-i\omega t}$.